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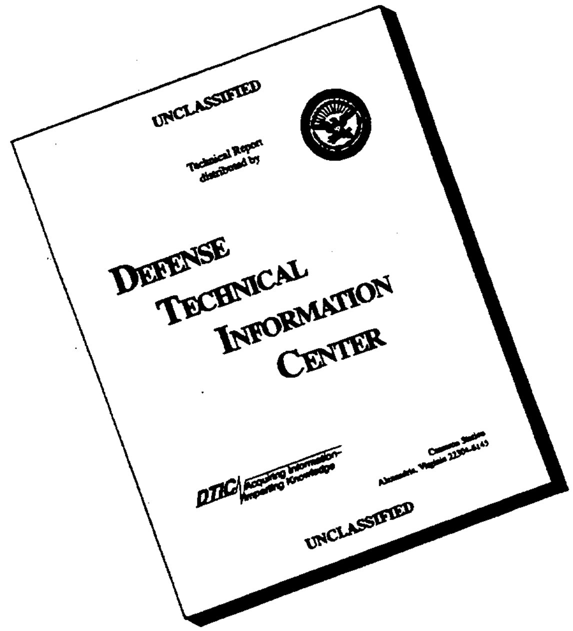

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FINAL REPORT

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THE NONLINEAR CONTROL THEORY OF COMPLEX MECHANICAL SYSTEMS

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ABSTRACT

This report provides a technical overview of research on the nonlinear control theory of complex mechanical systems, conducted under support of the U.S. Air Force Office of Scientific Research. The period of support has been June 15, 1990 through December 14, 1995. The research has had both theoretical and experimental components aimed at development of new approaches for controlling the motions of complex mechanical systems. The work has been principally concerned with classes of physical systems wherein the salient dynamical features cannot be understood in terms of linear models. Applications of interest include the control of molecular dynamics, microelectromechanisms, aerospace structures, rotating shafts, turbine dynamics, etc. Common elements in models of such systems form the basis of a general theory of control.

The principal focus of this report is on control strategies using tuned oscillatory forcing. Two papers in particular—"Stable Average Motions of Mechanical Systems Subject to Periodic Forcing" and "Energy Methods for Stability of Bilinear Systems with Oscillatory Inputs" provide a detailed account of some of our research. The broader scope of our research is indicated by the list of references which concludes the report.

Contents

The Nonlinear Control Theory of Complex Mechanical Systems

Grant AFOSR 90-0226

FINAL REPORT

Abstract

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1 Introduction and Report Summary

During the period of performance covered by this report, basic research concerned with the nonlinear control theory of complex mechanical systems was carried out at Boston University. Results of the research have been reported in *(i)* 28 technical papers appearing in the open literature (references [15-42] cited at the end of the report) and in *(ii)* numerous technical presentations at conferences, AFOSR contractors' review meetings, and lectures at educational institutions throughout the world. In May, 1993, two doctoral students who had been supported on the subject grant received Ph.D. degrees from Boston University. These were:

1. Daniel P. Martin, B.U. Ph.D. Thesis: "Mathematical Methods for Problems of Kinematic Redundancy in Robotics," and

2. Danbing Seto, B.U. Ph.D. Thesis: "Stabilization Problems in the Control of Super-Articulated Mechanical Systems."

The research covered by this report has been focused on several different problem areas:

(a) Robotic devices with redundant degrees of freedom (Refs. [16], [18],[22],[27-29],[32]).

Some of the early research sponsored under the subject grant dealt with planning motions for robotic mechanisms which simultaneously exhibit features of joint elasticity and kinematic redundancy ([27], [28], and [29]). It has been shown that for idealized models of mechanisms with certain congenial geometries, it is possible to plan motions which never store energy in their elastic degrees of freedom. Such motions follow acceleration constraints which are most naturally described in a coordinate system attached to some component of the mechanism. Results to date point to two lines for further study: *(i)* It is apparently only highly idealized (non-physical) models which admit the possibility of moving without storing energy. We wish to develop a rational basis for associating such models to actual systems with the aim of ascertaining whether "ideal" motions which result in no energy storage in the model also prescribe motions which minimize energy storage in the actual system. *(ii)* Current research by us and others seeks to develop control strategies which implement the motion plans that have been proposed.

(b) Modeling and control of the rotational dynamics of complex multibody systems and mixed structures (Refs. [20],[24],[33]).

Our early work in developing a control theory of mechanical systems concentrated on development of models of rotating mixed structures having multiple interconnected elastic and rigid components. Large mechanically complex earth satellites were the primary applications focus of this research. While there had been a great deal of

prior work on the control of space structures, there was little literature treating the application of ideas from geometric mechanics and nonlinear control theory to problems where such ideas were of apparent importance. In [3], we proposed a general Lagrangian formulation for the dynamics complex mechanisms capturing the general effects of inertial forces created by spatial rotations. Using Rayleigh type dissipation models, we developed a simple geometric formalism for modeling rotating viscoelastic systems. This theory was applied to analyze the dynamic behavior of a simple spatially rotating structure consisting of a rigid body with a viscoelastic beam attachment. Asymptotic steady state behavior of the model was determined, and a complete stability and bifurcation analysis was given for a closely related model having only one rotational degree of freedom. It is noted that parallel research on the modeling and analysis of mixed structure dynamics was also carried out with greater emphasis on Hamiltonian methods by Krishnaprasad, Marsden and others.

(c) Adaptive and robust control of super-articulated (underactuated) mechanical systems (Refs. [31],[34],[36]).

Some of the research cited in the references at the end of the report has been aimed at classifying multibody systems in terms of their topology and mechanical interconnections (joint types between bodies). (See, e.g., [31] and [36].) By assigning a certain digraph, which we call a *control flow diagram* (CFD), to each super-articulated mechanical system, we develop a formal notion of control *complexity* which provides information on how control design may be usefully approached. Certain "chain" systems, for instance, are shown to be feedback controllable using a backstepping controller design whose stability properties are easily deduced from the backstepping algorithm itself. More complex structures call for greater design effort, but the graphical methods that have been studied are useful in prescribing decoupling control laws which effectively transform certain structures into systems of chains to which our design methods apply directly. The CFD also points to a classification of mechanical systems, for which we may design stable adaptive feedback control laws to treat parametric uncertainties. (See [34].)

(d) Stability and control of mechanical systems using oscillatory actuator inputs (Refs. [17],[25],[26],[30],[37-41]).

Motivated by our early work on the control and bifurcation theory of equilibrium rotations in multibody systems (See [1-6], [11], and [24].), we have turned our attention to the broader issue of open-loop control designs tailored to exploiting key the intrinsic dynamical behavior in systems of interest. More specifically, we have been studying the use of oscillatory forcing to produce stable motions in a variety of mechanical systems, including rotating and vibrating kinematic chains as well as in the axial compressor problems discussed below. The mathematical mechanism underlying this approach to

stabilization involves the nonholonomic relationship between input and state variables in a fundamental way. (See [40] and [37] for details.) Using various extensions of classical averaging methods, it has been shown that the stabilizing effects of oscillatory forcing may be identified and analyzed in terms of an energy-like quantity call the *averaged potential.* (See [30] [40], and [37].) Under certain reasonable assumptions, forcing by periodic inputs produces dynamic responses in which states of the system are confined to neighborhoods of local minima of the averaged potential. While use of the averaged potential to analyze the stabilizing effect that can be produced by appropriately designed oscillatory forcing of conservative (Hamiltonian) mechanical systems highlights the geometric character of the phenomenon, we have also begun looking at this approach for influencing the stability characteristics of dissipative systems. The main body of this report consists of two papers we have published on the use of oscillatory forcing for stabilizing finite dimensional mechanical systems.

2 Stable Average Motion of Mechanical Systems Subject to Periodic Forcing

Stable Average Motions of Mechanical Systems Subject to Periodic Forcing

J. BAILLIEUL *

Abstract: There is a rapidly growing body of literature devoted to the study of anholonomy in the motions of nonlinear systems forced by periodic inputs. While a great deal of recent work has treated the case of systems without "drift", the emphasis in our treatment will be the controlled dynamics of mechanical systems in which "drift" terms play an important role. Our main application will be to controlling the dynamics of "super-articulated" mechanical systems—systems having fewer controls than configuration-space dimensions. Using classical averaging theory, it is shown that the stability of motion for such forced systems may be analyzed using energy methods together with the adroit introduction of dissipation into the models. For systems where the control inputs are applied directly only to cyclic coordinates, we define a simple but important quantity called the *averaged potential.* It is shown that under certain reasonable assumptions, forcing by periodic inputs produces dynamic responses which are confined to neighborhoods in the phase space associated with local minima of the averaged potential. We conclude with a discussion of linearization of the forced system about critical points of the averaged potential. While the critical points of the averaged potential may not correspond to equilibrium points of the original forced system, averaging theory implies that trajectories of the forced system will "hover" around such points. For an example problem in which there is an energy-like quantity which is conserved, it is shown by purely geometric means that Lyapunov stability may be deduced from a critical point analysis of the averaged potential. Hence, although the role of the averaged potential in our stability analysis is initially established by a classical averaging argument applied to an appropriate dissipative system, it is shown to be a geometric quantity which describes the effects of anholonomy and is independent of stability in the presence of dissipation.

1. INTRODUCTION

Consider a cart which is free to move on a frictionless track and to which there is attached a pendulum with a frictionless hinge whose axis is aligned with gravity. (Thus we have a two degree of freedom mechanical system in which there are no potential forces which influence the motion.)

*The author gratefully acknowledges the support of the U. S. Air Force through grant AFOSR-90-0226.

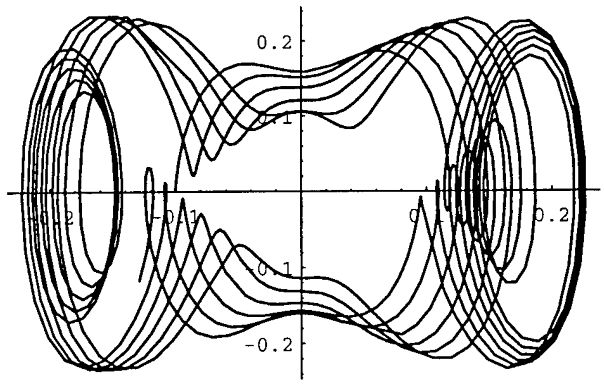

Figure 1: Phase portrait of (2) under periodic forcing $\dot{x} = \sin(9t)$.

Suppose the mass of the cart is m_c, and the pendulum consists of a rigid massless rod of length ℓ together with a tip of mass m_p. Assume further that we may apply a control force u to move the cart along its track, and no other exogenous forces are applied. Let x denote the distance (from some arbitrary reference point) that the cart has moved along the track, and let θ denote the angle of the pendulum with respect to the positive direction of the track. x and θ together comprise a system of generalized coordinates in terms of which we may write the dynamics of the system

$$(m_c + m_p)\ddot{x} - m_p \ell \ddot{\theta} \sin\theta - m_p \ell \cos\theta \dot{\theta}^2 = u \tag{1}$$

$$\ell\ddot{\theta} - \ddot{x}\sin\theta = 0. \tag{2}$$

With the single scalar control entering the dynamics in this way, the cart-pendulum system is a member of the class of mechanical systems which we have called *super-articulated.* (A *super-articulated mechanical systems* is a controlled mechanical system wherein the number of controls is strictly less than the dimension of the configuration space.) Broadly speaking, the goal of the present paper is to develop the elements of a control theory of super-articulated mechanical systems which will explain the behavior of (1)-(2) which is illustrated in Figure 1. This shows a phase portrait of (2) under periodic forcing $x = -\frac{1}{\alpha}\cos\alpha t$ with initial conditions $(\theta, \dot{\theta}) = (\cdot, \cdot)$. The point $(\theta, \dot{\theta}) = (0, 0)$ appears to be Lyapunov stable, and a proof of the fact that it is will emerge from our theory.

The remainder of the paper is organized as follows. In Section 2, we define what is meant by a *generalized mechanical control system.* This includes both classical finite-dimensional mechanical control systems (e.g. robots with rigid components) and "super-articulated" mechanical systems (e.g. robots with elastic joints) as special cases. Various examples are discussed. In Section 3, we apply classical averaging theory to the study of periodically forced generalized control systems. The *averaged potential* is defined in Section 4. This function is shown to be useful in characterizing the stability of motions in which classical potential forces interact with inertial forces created by the motion produced by the periodic forcing. While the properties of the *averaged potential* are illuminated using classical averaging theory in Section 4, Section 5 pursues a geometric analysis which shows that the averaged potential provides a simple measure of nonholonomic effects produced by periodic forcing of the class of systems under study.

This paper presents a geometrical framework within which the types of mechanics problems typically treated by classical averaging theory may also be solved using methods suggested by geometric nonlinear control theory. Advantages of this approach include a sharper characterization of dynamical behavior and more direct treatment of conservative (Hamiltonian) systems. Our geometric analysis in Section 5 is aimed at understanding the effects of anholonomy in certain periodically forced nonlinear systems, and in this work we have drawn inspiration from the recent work of Sussmann (e.g. [17], [18]), Sastry ([13]), their coworkers, and others ([10]). The problems we treat, however, differ significantly from "kinematic control" problems in that "drift" terms are present in the dynamics we treat, rendering both our methods and results quite different from those reported in [17], [18], and [13].

2. Generalized Mechanical Control Systems

Following Fliess ([7]) we state the following:

Definition 1 *A* generalized second-order control system *is prescribed by an evolution equation of the form*

$$\ddot{q}(t) = F(q(t), \dot{q}(t); u, \dot{u}, \dots, u^{(\alpha)})$$

where q takes values in a configuration manifold M and control-input curves $u(t)$ are bounded and piecewise analytic with values in a control-input manifold U.

NOTE: Assuming that the control u is only piecewise analytic means that the derivatives $\dot{u}, \dots$ may not be defined at all times t. This is a technicality which may be addressed by arbitrarily redefining $u(t)$ and it derivatives at points of ambiguity.

Since we shall be interested in local behavior, there is no loss of generality in assuming M and U are finite dimensional vector spaces of appropriate dimensions. As in [7], we further assume that for each $(q, \dot{q}) \in TM$, $F(q, \dot{q}; u, \dots, u^{(\alpha)})$ is a polynomial in u and its derivatives.

Definition 2 *A* generalized mechanical control system *is a generalized second order control system which has second-order polynomial dependence jointly on $\dot{x}$ and u and which has polynomial dependence on $\dot{u}$ which is at most first order. Hence the equations of a mechanical control system take the following form:*

$$\ddot{q} = g_0(q, \dot{q}) + g_1(q)\dot{u} + g_2(q, \dot{q})u + g_3(q)u^{[2]}. \tag{3}$$

where $u^{[2]}$ denotes the symmetric second tensor power of the vector u (See [6] for details!), and where for each q, $g_0(q, \dot{q})$ has at most second order polynomial dependence on $\dot{q}$, and $g_2(q, \dot{q})$ has at most first order polynomial dependence on $\dot{q}$.

Remark 1 To emphasize the polynomial dependence on $\dot{q}$ we write out the velocity terms in (3) as

$$g_0(q,\dot{q}) = g_0^0(q) + g_0^1(q)\dot{q} + g_0^2(q)\dot{q}^{[2]}, \tag{4}$$

$$g_2(q,\dot{q}) = g_2^0(q) + g_2^1(q)\dot{q}. \tag{5}$$

It is useful to examine some examples of mechanical control systems.

Example 1 (Cart-pendulum 1) Equation (2) may be viewed as a specialization of (3) in which θ is the configuration variable, $\ddot{x}$ is the control (i.e. $u = \ddot{x}$), $g_0 = g_1 = g_3 \equiv 0$, and g_2 is a function of θ alone with $g_2(\theta) = \sin\theta$.

Example 2 (Cart-pendulum 2) Equation (2) may also be viewed as a specialization of (3) in which θ is the configuration variable, $\dot{x}$ the control (i.e. $u = \dot{x}$), $g_0 = g_2 = g_3 \equiv 0$, and $g_1(\theta) = -\sin\theta$.

Remark 2 The structure (3) is sufficiently rich that there may be multiple ways in which to represent a particular physical system of interest. The choice of representation will be guided by the advantages afforded in applying the theory.

Example 3 (Lagrangian control systems) Let M be a real analytic differentiable manifold of dimension n. Following Nijmeijer and van der Schaft ([14]), we define a *Lagrangian control system* on M to be a dynamical system with inputs whose equations of motion are prescribed by applying the Euler-Lagrange operator to a function $L : TM \times U \to \mathbf{R}$, $L = L(q,\dot{q};u)$, whose dependence on the configuration q, the velocity $\dot{q}$, and the control input u is smooth. The set U in which the control input functions $u(\cdot)$ take values is assumed to be a closed convex subset of $\mathbf{R}^m$ which we further assume is symmetric with respect to the origin. (I.e., $u \in U \Rightarrow -u \in U$.)

A *nondegenerate* Lagrangian is one for which $\frac{\partial^2 L}{\partial \dot{q}^2} = (\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j})$ is nonsingular for all $(q,\dot{q};u) \in TM \times U$. The equations of motion for a control system arising from a nondegenerate Lagrangian may be written

$$\ddot{q} = -(\frac{\partial^2 L}{\partial \dot{q}^2})^{-1}(\frac{\partial^2 L}{\partial q \partial \dot{q}}\dot{q} + \frac{\partial^2 L}{\partial u \partial \dot{q}}\dot{u} - \frac{\partial L}{\partial q}).$$

Defined in this way, a Lagrangian control system is a generalized mechanical control system precisely when L has the proper polynomial dependence on the variable u.

A wide class of interesting mechanical control systems arise from Lagrangian functions of the form $L(q,\dot{q};u) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q) + q^T Bu$ where $q \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, and B is an $n \times m$ matrix.

Example 4 (Reduced-order symmetric Lagrangian systems) Consider a Lagrangian control system with configuration variables $(q_1, q_2) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$. The variable q_1 will be called *cyclic* if it does not enter into the Lagrangian when $u = 0$, i.e. if $\frac{\partial L}{\partial q_1}(q_1, q_2, \dot{q}_1, \dot{q}_2; 0) \equiv 0$. We shall be interested in Lagrangian control systems with cyclic variables in which it is precisely the cyclic variables which may be directly controlled. Specifically, we shall consider systems of the form

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2; u) = \frac{1}{2}(\dot{q}_1^T, \dot{q}_2^T)\begin{pmatrix} m(q_2) & A^T(q_2) \\ A(q_2) & M(q_2) \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} - V(q_2) + q_1^T u \tag{6}$$

where $\dim q_1 = n_1, \dim q_2 = n_2$, and m and M are symmetric positive definite matrices of dimension $n_1 \times n_1$ and $n_2 \times n_2$ respectively. To emphasize the distinguished role to be played by the cyclic variable, we write $v = \dot{q}_1$.

Applying the usual Euler-Lagrange operator to this function, we find the equations of motion are

$$\frac{d}{dt}(m(q_2)v + A(q_2)^T \dot{q}_2) = u \tag{7}$$

$$\ddot{q}_2 + M(q_2)^{-1}(\Gamma(q_2, \dot{q}_2) + \frac{\partial V}{\partial q_2} + A(q_2)\dot{v} + \mathcal{G}(q_2, \dot{q}_2)v - \frac{1}{2}v^T \frac{\partial m}{\partial q_2} v) = 0 \tag{8}$$

where the k-th component $(1 \le k \le n_2)$

$$\Gamma(q_2, \dot{q}_2)_k = \frac{1}{2}\sum_{i,j}(\frac{\partial M_{ki}}{\partial q_2^j} + \frac{\partial M_{kj}}{\partial q_2^i} - \frac{\partial M_{ij}}{\partial q_2^k})\dot{q}_2^i \dot{q}_2^j,$$

and the k-th component

$$[\mathcal{G}(q_2, \dot{q}_2)v]_k = \sum_{i=1}^{n_2}\sum_{j=1}^{n_1}(\frac{\partial a_{kj}}{\partial q_2^i} - \frac{\partial a_{ij}}{\partial q_2^k})\dot{q}_2^i v_j.$$

Noting that u and v have the same dimensions and that $m(q)$ is nonsingular for all q, it is an immediate consequence of existence and uniqueness theorems for ordinary differential equations that for any continuous trajectory $q_2(t)$, any smooth trajectory $v(\cdot)$ may be generated by (7) by means of an appropriate choice of control input $u(\cdot)$. This means that in principle, it is immaterial in controlling (7)-(8) whether we view u as the control with v being determined by integrating the differential equation (7) or take v to be the control with u then simply determined by (7). If v is viewed as the control input, (8) clearly has the form of a mechanical control system (3). It is interesting to note that (8) itself prescribes the dynamics of a Lagrangian control system with associated *reduced Lagrangian*

$$\hat{L}(q_2, \dot{q}_2; v) = \frac{1}{2}\dot{q}_2^T M(q_2)\dot{q}_2 + \dot{q}_2^T A(q_2)v - V_a(q; v),$$

where V_a is the *augmented potential* defined by $V_a(q; v) = V(q) - \frac{1}{2}v^T m(q_2)v$. The terms $\mathcal{G}(q_2, \dot{q}_2)v$ which are linear in the generalized velocities are called *gyroscopic terms.* Many physical systems admit this type of reduction, including the cart-pendulum system (1)-(2) and the rotating kinematic chain systems studied in [2] and [12]. For more details on the role played by gyroscopic terms in the dynamics of mechanical systems, the reader is referred to [3] and [9].

3. Averaging and the Stability of Motion

Although many interesting physical systems may be described by models of the form (3), the control theory of such systems is presently incomplete. While it is generally unclear where to turn for inspiration in classical geometric nonlinear control theory, recent work by Murray and Sastry, [13], suggests that the geometry of noncommuting vectorfields may be systematically exploited by using carefully selected time-periodic inputs to induce desired motions. (See also related work by Sussmann, [18].) Such results are not immediately applicable to systems of the form (3), however, and it is our goal in the remainder of the paper to develop the appropriate theory. In the present section, our principal mathematical tool will be classical averaging theory.

Suppose the input $u(\cdot)$ appearing in (3) is a periodic function of time with fundamental period $T > 0$. To put (3) into a form to which standard results in averaging theory may be applied, we convert to first order form and rewrite the equations in the time scale $\tau = t/T$. (Cf. [8], p. 166.) Letting $y(\tau) = x(t)$, the resulting differential equations take the form

$$\frac{d}{d\tau}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = T \cdot \begin{pmatrix} y_1 \\ g_0(y_1, y_2) + g_1(y_1)\dot{u}_s + g_2(y_1, y_2)u_s + g_3(y_1)u_s^{[2]} \end{pmatrix}, \tag{9}$$

where $\dot{u}_s(\tau) = \dot{u}(T\tau)$, $u_s(\tau) = u(T\tau)$. As the frequency of the forcing is varied, we find the amplitude of $\dot{u}_s$ grows like $\frac{1}{T}$. (Think of varying the frequency of a sinusoid.) Hence, averaging theory does not immediately apply unless $g_1 = 0$. Fortunately, a simple time-varying coordinate transformation eliminates the dependence on the derivative of the input.

Definition 3 *Given the mechanical system (3), the* L-transformation *is defined by the equation*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} - g_1(q)u \end{pmatrix}. \tag{10}$$

Important features of this transformation are given in the following.

Proposition 1 (i) *The L-transformation is invertible.* (ii) *In (x_1, x_2)-coordinates (3) is rendered*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 + g_1(x_1)u \\ G_0(x_1, x_2) + G_1(x_1, x_2)u + G_2(x_1)u^{[2]} \end{pmatrix}, \tag{11}$$

where $G_0(x_1, x_2) = g_0(x_1, x_2)$, and G_1 has first order polynomial dependence on x_2. The G_i's are given explicitly by equating coefficients in

$$G_0(x_1, x_2) + G_1(x_1, x_2)u + G_2(x_1)u^{[2]} \equiv g_0(x_1, x_2 + g_1(x_1)u) + g_2(x_1, x_2 + g_1(x_1)u)u$$

$$+ g_3(x_1)u^{[2]} - \left[\frac{\partial}{\partial x_1}(g_1(x_1)u)\right](x_2 + g_1(x_1)u). \tag{12}$$

Proof: It is easy to see that the inverse L-transformation is given by

$$\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + g_1(x_1)u \end{pmatrix}.$$

Using the formulas for the forward and inverse L-transformations, it is a straightforward calculation to show that in terms of (x_1, x_2)-coordinates (3) is rendered as (11).

Remark 3 In the case that our system is given by (8), the L-transformation is the ordinary Legendre transformation provided that $M(q) = I$.

It is not difficult to see that (11) inherits a polynomial dependence on u and x_2 which is similar to (3). The main feature of the L-transformation is that it eliminates the dependence on the derivative of the input $\dot{u}$. We shall next see that if $u(\cdot)$ is a high frequency periodic input, then by appropriately scaling time the equations (11) may be rendered in a form to which averaging theory applies.

Indeed, suppose $u(\cdot)$ is a bounded piecewise continuous $\mathbf{R}^m$-valued function defined on $(-\infty, \infty)$ which is periodic of fundamental period $T > 0$. Write

$$u_j(t) = \sum_{k=-\infty}^{\infty} c_{jk} e^{\frac{2\pi k}{T} it}, \tag{13}$$

and let

$$\bar{u} = \frac{1}{T} \int_0^T u(t)\, dt$$

and

$$\sigma^2 = \frac{1}{T} \int_0^T u(t)^{[2]}\, dt.$$

The corresponding averaged version of (11) is

$$\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_2 + g_1(\xi_1)\bar{u} \\ G_0(\xi_1, \xi_2) + G_1(\xi_1, \xi_2)\bar{u} + G_2(\xi_1)\sigma^2 \end{pmatrix}. \tag{14}$$

It is interesting to note that if we change the frequency of the input function $u(\cdot)$, the form of the averaged equations remains unchanged. Specifically, writing $u_\lambda(t) = u(\lambda t)$ we see that u_λ has period $T_\lambda = T/\lambda$ and the same mean $\bar{u} = T_\lambda^{-1} \int_0^{T_\lambda} u_\lambda\, dt$ and r.m.s. value $\sigma^2 = T_\lambda^{-1} \int_0^{T_\lambda} u_\lambda^{[2]}\, dt$. In the next theorem, we shall view frequency as a control parameter.

Suppose (ξ_1^0, ξ_2^0) is an equilibrium solution of (14). Classical averaging theory yields the following.

Theorem 1 *Let $u(\cdot)$ be an $\mathbf{R}^m$-valued bounded piecewise continuous periodic function of period $T > 0$ whose j-th entry is given by (13) for $1 \le j \le m$. Let (ξ_1^0, ξ_2^0) be an equilibrium solution of (14), and suppose that when evaluated at (ξ_1^0, ξ_2^0) the $2n \times 2n$-matrix*

$$\begin{pmatrix} \frac{\partial g_1}{\partial \xi_1}\bar{u} & I \\ \frac{\partial G_0}{\partial \xi_1} + \frac{\partial G_1}{\partial \xi_1}\bar{u} + \frac{\partial G_2}{\partial \xi_1}\sigma^2 & \frac{\partial G_0}{\partial \xi_2} + \frac{\partial G_1}{\partial \xi_2}\bar{u} \end{pmatrix}$$

has all its eigenvalues in the left half-plane. Then for every $\epsilon > 0$ there is a $\delta > 0$ and λ_0 such that for all $\lambda > \lambda_0$, if (11) is forced by the input $u_\lambda(t) = u(\lambda t)$, we find the solution $(x_1(t), x_2(t))$ of (11) is such that $\|x(t) - \xi^0\| < \epsilon$ provided $\|x(0) - \xi^0\| < \delta$.

Proof: Given ϵ, δ, and λ as in the hypothesis of the theorem, we shall write both (11) and (14) in the time-scale $\tau = t/T_\lambda$ where $T_\lambda = \frac{T}{\lambda}$ is the fundamental period of $u_\lambda(\cdot)$. Letting $y(\tau) = x(t)$, (11) becomes

$$\frac{d}{d\tau}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = T_\lambda \cdot \begin{pmatrix} y_2 + g_1(y_1)u_\lambda(\tau T_\lambda) \\ G_0(y_1, y_2) + G_1(y_1, y_2)u_\lambda(\tau T_\lambda) + G_2(y_1)u_\lambda(\tau T_\lambda)^{[2]} \end{pmatrix}. \tag{15}$$

(Cf. (9).) Similarly, letting $\zeta(\tau) = \xi(t)$, (14) becomes

$$\frac{d}{dt}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = T_\lambda \cdot \begin{pmatrix} \zeta_2 + g_1(\zeta_1)\bar{u} \\ G_0(\zeta_1, \zeta_2) + G_1(\zeta_1, \zeta_2)\bar{u} + G_2(\zeta_1)\sigma^2 \end{pmatrix}. \tag{16}$$

Note that $\tilde{u}(\tau) = u_\lambda(\tau T_\lambda)$ is a periodic function of period 2π independent of λ.

Now (15) is a nonautonomous equation to which classical averaging theory may be applied. (Specifically, see [5], p. 497.) Under our hypothesis, the eigenvalues of the Jacobian of the right hand side vector field (16), evaluated at (ξ_1^0, ξ_2^0), all have negative real parts. Hence for all λ sufficiently large (and $T_\lambda = T/\lambda$ correspondingly small), there is a unique periodic (period 2π) solution $(y_1(\tau), y_2(\tau))$ to (15) defined on the entire infinite interval $(-\infty, \infty)$ such that $\|y(\tau) - \xi^0\|_\infty < \eta(\lambda)$ for $-\infty < \tau < \infty$ with $\eta(\lambda) \to 0$ as $\lambda \to \infty$. (See [5].) Moreover, if $\tilde{y}(\tau)$ is any solution to (15) which is sufficiently close to ξ^0 at some time $\tau = \tau_0$, then $\|\tilde{y}(\tau) - y(\tau)\| \to 0$ exponentially as $\tau \to \infty$. There is no loss of generality in assuming $\tau_0 = 0$.

To complete the proof of the theorem, let λ be sufficiently large that $\eta(\lambda) < \frac{\epsilon}{2}$. Moreover, let τ_1 be such that for all $\tau > \tau_1$ we have $\|\tilde{y}(\tau) - y(\tau)\| < \epsilon/2$. Let $\delta > 0$ be so small that if $\|y(0) - \tilde{y}(0)\| < \delta$, then $\|y(\tau) - \tilde{y}(\tau)\| < \epsilon/2$ on $0 \le \tau < \tau_1$. Now λ has been chosen large enough to ensure that $\eta(\lambda) < \frac{\epsilon}{2}$, and hence $\|y(\tau) - \xi^0\|_\infty < \frac{\epsilon}{2}$. Thus if $\|\tilde{y}(0) - \xi^0\| < \frac{\delta}{2}$, we have

$$\begin{aligned} \|y(0) - \tilde{y}(0)\| &\le \|y(0) - \xi^0\| + \|\tilde{y}(0) - \xi^0\| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta. \end{aligned} \tag{17}$$

Hence $\|\tilde{y}(\tau) - y(\tau)\| < \frac{\epsilon}{2}$ for all $\tau \ge 0$, and thus $\|\tilde{y}(\tau) - \xi^0\| < \epsilon$. Since x and y are related by a change of time scale, this proves the theorem. □

Remark 4 Under the hypothesis of the theorem, we have proved that solutions of (11) will execute motions confined to a neighborhood of (ξ_1^0, ξ_2^0). Moreover, as the frequency of the input function $u(\cdot)$ becomes larger, this neighborhood to which the motions are confined shrinks. We may relate these observations to motions of (3). The variables $(q, \dot{q})$ are related to (x_1, x_2) by means of the inverse

L-transformation, and it is not difficult to see that trajectories expressed in these coordinates may be expected to execute motions in a neighborhood of the point

$$\begin{pmatrix} q_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi_1^0 \\ \xi_2^0 + g_1(\xi_1^0)\bar{u} \end{pmatrix}.$$

Since the trajectory $(q, \dot{q})$ is related to the trajectory (x_1, x_2) by means of this time-varying coordinate change, the size of the neighborhood to which the $(q, \dot{q})$-motions are confined does not necessarily shrink as the frequency of the forcing is increased.

Remark 5 It is important to note that this theorem does not actually deal with stability of equilibria for systems of the form (11) since (ξ_1^0, ξ_2^0) needs only to be an equilibrium (stable) for the averaged system (14). It is easy to find examples (See below!) where $(x_1, x_2) = (\xi_1^0, \xi_2^0)$ is not an equilibrium of (11) and yet (11), under the hypotheses of the theorem, executes motions confined to a neighborhood of (ξ_1^0, ξ_2^0).

Example 5 (Example 1 reprise) The averaged equations (14) in this case take the form

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_2 \\ 0 \end{pmatrix}.$$

Clearly the hypotheses of the theorem are not met, and no conclusion can be drawn. A possible way to further analyze this system is to pass to higher order averaging approximations as discussed in [5]. Equivalently, we may also pass to the representation of the system given in Example 2.

Example 6 (Example 2 reprise) For this representation, the averaged equations (14) take the form

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_2 + \bar{u}\sin\xi_1 \\ -\bar{u}\cos\xi_1 \cdot \xi_2 - \sigma^2 \sin\xi_1 \cos\xi_1 \end{pmatrix}.$$

Taking $x(t) = -\frac{1}{\alpha}\cos\alpha t$, we have $u(t) = \sin\alpha t$, and $\bar{u} = 0$, $\sigma^2 = \frac{1}{2}$. $(\xi_1^0, \xi_2^0) = (0,0)$ is clearly an equilibrium solution of the averaged equations. The linearized version of this differential equation at this equilibrium has eigenvalues on the imaginary axis, and hence the hypothesis of Theorem 1 is again not satisfied. We may proceed, however, by appealing to physical considerations.

Returning to the description of the cart-pendulum system in the introduction, we note that if a rate-dependent damping term, $d\dot{\theta}$, is included in the model, (2) becomes

$$\ell\ddot{\theta} + d\dot{\theta} - \ddot{x}\sin\theta = 0.$$

In terms of the general representation (3), $u = \dot{x}$, $g_0(\theta, \dot{\theta}) = -d\dot{\theta}$, $g_1(\theta) = \sin\theta$, $g_2 \equiv 0$, and $g_3 \equiv 0$. The corresponding averaged equations are

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \begin{pmatrix} \xi_2 + \bar{u}\sin\xi_1 \\ -d(\xi_2 + \bar{u}\sin\xi_1) - \bar{u}\cos\xi_1 \cdot \xi_2 - \sigma^2 \sin\xi_1 \cos\xi_1 \end{pmatrix}.$$

Linearizing about the equilibrium $(\xi_1^0,\xi_2^0)=(0,0)$ yields the variational equation

$$\frac{d}{dt}\begin{pmatrix}\delta\xi_1\\ \delta\xi_2\end{pmatrix}=\begin{pmatrix}\bar u & 1\\ -d\bar u-\sigma^2 & -d-\bar u\end{pmatrix}\begin{pmatrix}\delta\xi_1\\ \delta\xi_2\end{pmatrix},$$

the coefficient matrix of which has characteristic polynomial $p(s)=s^2+ds+\sigma^2-\bar u^2$. Noting that $\sigma^2-\bar u^2>0$, the eigenvalues are seen to lie in the open left half plane. Hence, by our theorem, the forced system can be expected to execute stable motions in a neighborhood of the origin.

This observation is consistent with what we would expect to observe if viscous damping were added to the dynamics of a Lyapunov stable system. It remains to show, however, that the origin $(\theta,\dot\theta)=(0,0)$ is Lyapunov stable for the periodically forced system (2). While the above analysis does not directly imply this, it suggests a general approach to stability analysis for conservative systems. The idea is to add Rayleigh dissipation (along the lines suggested, say, in [1]) and then assess the spectral stability of the resulting averaged system. Similar ideas have recently been explored in [4] in connection with the energy-momentum method. We shall provide a brief general sketch.

For systems of the form (3) dissipation enters through the terms $g_0^1(q)\dot q$ (see (4)). Indeed, we shall assume that $g_0^1(q)$ is a symmetric negative definite matrix so that $-\dot q^T g_0^1(q)\dot q$ is the usual Rayleigh dissipation function. We shall be interested in comparing systems of the form (3) which differ from one another precisely according to whether the Rayleigh dissipation function is absent or present. Accordingly, we rewrite the equation of interest, (3), in the form

$$\ddot q=g_0^0(q)+\epsilon g_0^1(q)\dot q+g_0^2(q)\dot q^{[2]}+g_1(q)\dot u+g_2(q,\dot q)u+g_3(q)u^{[2]}, \tag{18}$$

where $\epsilon=0,1$ depending on whether damping is absent or present. Applying the L-transformation (10) and averaging, we obtain from (18)

$$\frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}\xi_2+g_1(\xi_1)\bar u\\ \epsilon g_0^1(\xi_1)(\xi_2+g_1(\xi_1)\bar u)+G_0(\xi_1,\xi_2)+G_1(\xi_1,\xi_2)\bar u+G_2(\xi_1)\sigma^2\end{pmatrix}, \tag{19}$$

where the G_i's are related to the g_i's via equation (12) <u>with $\epsilon=0$</u> in $g_0(\xi_1)$. Suppose (ξ_1^0,ξ_2^0) is an equilibrium for this differential equation; i.e. $\xi_2^0+g_1(\xi_1^0)\bar u=0$ and $G_0(\xi_1^0,\xi_2^0)+G_1(\xi_1^0,\xi_2^0)\bar u+G_2(\xi_1^0)\sigma^2=0$. Then we have the following.

Theorem 2 *For the conservative version of (19) (i.e. when $\epsilon=0$), let us suppose that for some admissible control input $u(\cdot)$ (i.e. for some bounded piecewise analytic u) and corresponding $\bar u$, σ there is a function E of the form*

$$E(\xi_1,\xi_2;\bar u,\sigma)=\frac{1}{2}\|\xi_2+g_1(\xi_1)\bar u\|^2+V(\xi_1;\bar u,\sigma)$$

which (i) *has a strict local minimum at the equilibrium (ξ_1^0,ξ_2^0), and* (ii) *is invariant under the motion of (19). Then for the dissipative version of (19) (i.e. when we set $\epsilon=1$) the equilibrium (ξ_1^0,ξ_2^0) is asymptotically stable.*

Proof: Differentiating E along trajectories of (19) we obtain

$$\dot{E} = \epsilon(\xi_2 + g_1(\xi_1)\bar{u})^T g_0^1(\xi_1)(\xi_2 + g_1(\xi_1)\bar{u}).$$

Here E serves as a Lyapunov function, and the right hand side of the equation is nonpositive and has a strict local maximum value of 0 at (ξ_1^0, ξ_2^0). By Lyapunov's second method, this equilibrium solution is asymptotically stable. □

Remark 6 The function E plays the role of total energy for the averaged system (19). This role may be emphasized by applying the *averaged* inverse L-transformation

$$\begin{pmatrix} \bar{q} \\ \dot{\bar{q}} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 + g_1(\xi_1)\bar{u} \end{pmatrix}$$

and expressing E in terms of the transformed variables $(\bar{q}, \dot{\bar{q}})$:

$$E(\bar{q}, \dot{\bar{q}}; \bar{u}, \sigma) = \frac{1}{2}\|\dot{\bar{q}}\|^2 + V(\bar{q}; \bar{u}, \sigma).$$

Corollary 1 *For the conservative version of (19) (i.e. when $\epsilon = 0$), suppose that for some admissible control input $u(\cdot)$ and corresponding $\bar{u}$, σ there is a function $E(\xi_1, \xi_2; \bar{u}, \sigma)$ as in Theorem 2 which has a strict local minimum at the equilibrium (ξ_1^0, ξ_2^0) and which is invariant under (19). Then for every $\eta > 0$ there is a $\delta > 0$ and λ_0 such that for all $\lambda > \lambda_0$ if the corresponding dissipative version of (11) (derived by applying the L-transformation to (18) with $\epsilon = 1$) is forced by $u_\lambda(t) = u(\lambda t)$, the solution $x(t) = (x_1(t), x_2(t))$ is such that $\|x(t) - \xi^0\|_\infty < \eta$ provided $\|x(0) - \xi^0\| < \delta$.*

Proof: This corollary follows directly from Theorems 1 and 2 and standard Lyapunov stability theory. □

Remark 7 In Lagrangian and Hamiltonian mechanics, the most obvious choice of function which is invariant under the dynamics of the system is the total energy. Thus to analyze mechanical systems with inputs of the form (3), it is of interest to ask whether there is any systematic procedure for relating the corresponding averaged system (14) to a Hamiltonian. (Because inputs are present, the usual total energy is not conserved.) In the next section, it will be shown that the reduced-order symmetric Lagrangian control systems of Example 4 retain a structure which may be related via the Legendre transformation to a Hamiltonian system. Averaging this system leads to another Hamiltonian system, so that stability analysis along the lines we have indicated may be carried out. Moreover, the analysis will be considerably simplified for this class of systems because, as we shall show, there is a natural block-diagonalizing coordinate transformation such that in the new coordinates the total averaged energy (=the averaged Hamiltonian) may be expressed as the sum of a positive definite averaged kinetic energy function and an averaged potential function. Equilibrium points of the averaged system will be found as critical points of the averaged potential, and the stable equilibria of interest in our analysis will be local minima of the averaged potential.

4. ENERGY METHODS FOR STABILITY AND THE *Averaged Potential*

Consider a symmetric Lagrangian control system of the form (6) where q_1 is a cyclic variable as defined in Example 4. Write $v = \dot{q}_1$. As in the preceding section, we view v as a control, and note that (8) arises from applying the Euler-Lagrange operator to the reduced Lagrangian

$$\hat{L}(q,\dot{q};v) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \dot{q}^T A(q)v - V_a(q;v), \tag{20}$$

where $V_a(q;v) = V(q) - \frac{1}{2}v^T m(q)v$ is the *augmented potential.*

To render the equations of motion in a form suitable for averaging, we apply the Legendre transformation $H(q,p;v) = p\cdot\dot{q} - \hat{L}(q,\dot{q};v)$ to obtain

$$H(q,p;v) = \frac{1}{2}(p - vA)^T M^{-1}(p - vA) + V_a(v,q). \tag{21}$$

The equations of motion are written in the usual way as

$$\dot{q} = M^{-1}(p - vA) \tag{22}$$

$$\dot{p} = -\frac{\partial}{\partial q}[\frac{1}{2}(p - vA)^T M^{-1}(p - vA) + V_a(v,q)]. \tag{23}$$

To pursue averaging theory, we replace all coefficients in (22)-(23) by their time-averages. Assuming $v(\cdot)$ is bounded, piecewise continuous, and periodic of period $T > 0$, we may write

$$v(t) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{2\pi k}{T}it}. \tag{24}$$

(Cf. (13).) (22)-(23) contain terms of order not greater than two in v, and averaging the coefficients we obtain

Proposition 2 *Suppose $v(\cdot)$ is given by (24). Then if all coefficients in (22)-(23) are replaced by their time averages, the resulting averaged system is Hamiltonian with corresponding Hamiltonian function*

$$\bar{H}(q,p) = \frac{1}{2}(p - A(q)\bar{v})^T M(q)^{-1}(p - A(q)\bar{v}) + V_A(q),$$

where

$$V_A(q) = V(q) + \frac{1}{2}(\Sigma(q) - \bar{v}^T A(q)^T M(q)^{-1} A(q)\bar{v})$$

is the averaged potential *with*

$$\bar{v} = \frac{1}{T}\int_0^T v\,dt,$$

and

$$\Sigma(q) = \frac{1}{T}\int_0^T v(t)^T(A(q)^T M(q)^{-1}A(q) - m(q))v(t)\,dt.$$

□

The averaged Hamiltonian $\bar{H}(q,p)$ may be interpreted as an averaged total energy. Writing this is terms of the variables q and $\dot{q}$, where $\dot{q} = M^{-1}(p - A\bar{v})$ the *averaged energy* is

$$\bar{E}(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + V_A(q).$$

Equilibrium solutions to the averaged Hamiltonian system

$$\dot{q} = \frac{\partial \bar{H}}{\partial p} \qquad \dot{p} = -\frac{\partial \bar{H}}{\partial q} \tag{25}$$

will be of the form $(q,\dot{q}) = (q_0, 0)$. We have the following:

Proposition 3 *Let $(q,p) = (q_0,p_0)$ be an equilibrium solution of the averaged Hamiltonian system (25). (q_0,p_0) is a strict local minimum of the total averaged energy $\bar{H}(q,p)$ if and only if q_0 is a strict local minimum of the* averaged potential $V_A(q)$.

Proof: The corresponding equilibrium in $(q,\dot{q})$-coordinates is $(q_0,0)$. (q_0,p_0) is a strict local minimum of $\bar{H}(q,p)$ if and only if $(q_0,0)$ is a strict local minimum of $\bar{E}(q,\dot{q})$. But the Hessian of $\bar{E}$ evaluated at $(q_0,0)$ has block diagonal form

$$\begin{pmatrix} \frac{\partial^2 V_A}{\partial q^2}(q_0) & 0 \\ 0 & M(q_0) \end{pmatrix}.$$

The proposition follows from this observation. □

To discuss stability, we introduce the Rayleigh dissipation function $\dot{q}^T D(q)\dot{q}$ where for each q, $D(q)$ is an $n \times n$ positive definite matrix. Starting from the reduced Lagrangian (20), we are interested in the behavior of the system

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial \dot{q}} - \frac{\partial \hat{L}}{\partial q} + D(q)\dot{q} = 0. \tag{26}$$

The Hamiltonian form of (26) is given by

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} - D(q)M(q)^{-1}(p - A(q)v), \end{aligned} \tag{27}$$

where H is as in (21). Averaging these equations, we obtain the system

$$\dot{q} = \frac{\partial \bar{H}}{\partial p}$$

$$\dot{p} = -\frac{\partial \bar{H}}{\partial q} - D(q)M(q)^{-1}(p - A(q)\bar{v}), \tag{28}$$

where the averaged Hamiltonian $\bar{H}$ is given in Proposition 2. The role of the averaged potential in assessing asymptotic stability is summarized as follows.

Theorem 3 *Consider a Lagrangian control system (20) with dissipation entering through a quadratic dissipation function $\dot{q}^T D(q)\dot{q}$ as described above. Let $v(\cdot)$ be given by (24), let $v_\lambda(t) = v(\lambda t)$, and for each λ, consider the averaged potential $V_A(q;\lambda)$ corresponding to $v_\lambda(\cdot)$ as in Proposition 2. Suppose further that there is a λ_0 and a one-parameter family of configurations q_λ depending smoothly on λ with the property that for each $\lambda > \lambda_0$ the corresponding q_λ is a strict local minimum of $V_A(q;\lambda)$. Then there is a $\lambda_1 \geq \lambda_0$ such that for each $\epsilon > 0$ and $\lambda > \lambda_1$ there is a $\delta = \delta(\epsilon, \lambda)$ with the property that if (27) is forced by $v_\lambda(\cdot)$, the solution $(q(t), \dot{q}(t))$ satisfies*

$$\|(q, \dot{q}) - (q_\lambda, 0)\|_\infty < \epsilon$$

provided

$$\|(q(0), \dot{q}(0)) - (q_\lambda, 0)\| < \delta$$

Proof: Let $\epsilon > 0$ be given, and let $v(\cdot)$ be defined by (24) with corresponding $v_\lambda(\cdot)$ as in the statement of the theorem. Let $\bar{H}_\lambda$ be the corresponding averaged Hamiltonian. If we let q, $\dot{q} = M(q)^{-1}(p - A(q)\bar{v})$ be the corresponding Lagrangian coordinates of the averaged system, $\bar{H}_\lambda$ may be rewritten in these coordinates as

$$\bar{E}_\lambda(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + V_A(q;\lambda).$$

In $(q, \dot{q})$-coordinates, (28) is rendered as

$$\frac{d}{dt}(M(q)\dot{q}) - \frac{\partial}{\partial q}[\frac{1}{2}\dot{q}^T M(q)\dot{q} + V_A(q;\lambda)] + D(q)\dot{q} = 0. \tag{29}$$

Differentiating $\bar{E}_\lambda(q, \dot{q})$ along trajectories of this system, we find

$$\frac{d\bar{E}_\lambda}{dt} = -\dot{q}^T D(q)\dot{q}.$$

$D(q)$ is positive definite for each q, and since $(q, \dot{q}) = (q_\lambda, 0)$ is a strict local minimum of $\bar{E}_\lambda$, it follows that $(q_\lambda, 0)$ is an asymptotically stable equilibrium of the averaged system (29).

It is now more or less straightforward to apply Theorem 1 to determine the behavior of (26). To do this, apply the L-transformation to the variables $(q, \dot{q})$. Now the L-transformation is related to the Legendre transformation by a left multiplication by $\mathrm{diag}(I, M(q)^{-1})$. Hence the asymptotically stable equilibrium solution, $(q_\lambda, 0)$, of (29) corresponds to an asymptotically stable equilibrium solution of (14). It thus follows that (26) executes motions in a neighborhood of $(q_\lambda, 0)$. □

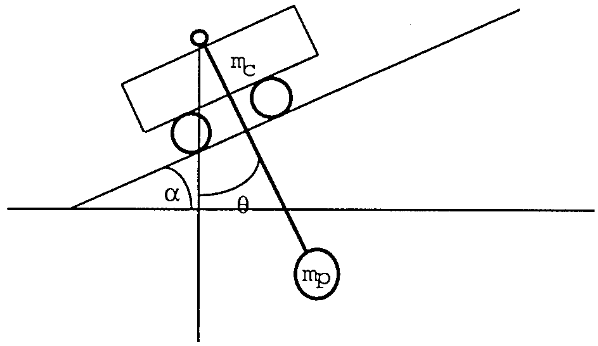

Figure 2: A cart-pendulum system on an inclined track.

Remark 8 The form of the averaged system (29) may be determined by examining the form of the system (26) from which it is derived. Specifically, in passing from (26) to (29), terms of the form

$$\sum_{j=1}^{m}\Big[\dot{v}_j A_j(q) + v_j(\frac{\partial A_j}{\partial q} - \frac{\partial A_j}{\partial q}^T)\dot{q}\Big] \tag{30}$$

are replaced by

$$\frac{1}{2}\frac{\partial}{\partial q}[\Sigma(q) - \bar{v}^T(A(q)^T M(q)^{-1}A(q) - m(q))\bar{v}]. \tag{31}$$

Example 7 (Example 6 reprise.) It is of interest to consider an extension of the cart-pendulum problem which serves to illustrate the power of the theory developed so far. Consider a cart to which there is a pendulum attached moving along a track in a uniform gravitational field. Suppose the track is inclined at an angle α with respect to the horizontal. As in the introduction, the pendulum consists of a point-mass attached by a massless link of length ℓ to the cart which has mass m_c. The coupled equations of motion for this system are

$$(m_c + m_p)\ddot{r} + m_p\ell\cos(\theta-\alpha)\ddot{\theta} - m_p\ell\sin(\theta-\alpha)\dot{\theta}^2 + (m_c+m_p)g\sin\alpha = u, \tag{32}$$

and

$$\ell\ddot{\theta} + \ddot{r}\cos(\theta-\alpha) + g\sin\theta = 0, \tag{33}$$

which reflect the assumption that control of this system is by pushing the cart along its inclined track. As in our earlier analysis, we may assume that the motion $r(t)$ can be specified, and taking $v(t) = \dot{r}(t)$, we find that (33) is derivable from the reduced Lagrangian

$$\hat{L}(\theta,\dot{\theta};v) = \frac{1}{2}\ell\dot{\theta}^2 + v\dot{\theta}\cos(\theta-\alpha) + g\cos\theta. \tag{34}$$

Assume $v(t)$ is given as in (24). Referring to the formulas in Proposition 2, we write the averaged potential corresponding to $\hat{L}$:

$$V_A(\theta) = -g\cos\theta + \frac{1}{2\ell}\cos^2(\theta-\alpha)\,(\sigma^2 - \bar{v}^2)$$

where

$$\bar{v} = \frac{1}{T}\int_0^T v(t)\,dt$$

and

$$\sigma^2 = \frac{1}{T}\int_0^T v(t)^2\,dt.$$

Stable equilibria of the averaged system correspond to local minima of V_A. One interesting special case occurs when $g = 0$ (i.e. the gravitational field is removed), $\alpha = 0$, and $v(t) = \sin\omega t$. This corresponds to the example treated in the introduction. Then $\bar{v} = 0$ and $\sigma^2 = \frac{1}{2}$, and $\theta = \pm\frac{\pi}{2}$ are local minima of V_A. If there is any dissipation in the hinge on the pendulum, then according to Theorem 3, the pendulum may be expected to execute motions confined to a neighborhood of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. Another interesting special case is to take $\alpha = \frac{\pi}{2}$ with nonzero g and $r(t) = -\cos(\omega t)$. (The cart executes a simple periodic vertical motion along its track.) Then $v(t) = \omega\sin(\omega t)$, $\bar{v} = 0$, and $\sigma^2 = \frac{\omega^2}{2}$ so that the critical points of the averaged potential are solutions of

$$g\sin\theta + \frac{\omega^2}{2\ell}\sin\theta\,\cos\theta = 0.$$

This equation has solutions $\theta = 0, \pi$ for all values of the parameters, and when $\omega^2 \geq 2g\ell$ there are also solutions $\theta = \arccos(-2\frac{g\ell}{\omega^2})$. (We shall restrict $\theta \in (-\pi, \pi]$.) Evaluating the second derivative of V_A at each of these critical points, we find that $\theta = 0$ is always a local minimum, and $\theta = \pi$ is a local maximum when $\omega^2 < 2g\ell$, but it becomes a local minimum when $\omega^2 > 2g\ell$. It follows from Theorem 3, that there is an $\omega_0 \geq 2g\ell$ such that for all $\omega > \omega_0$ the pendulum can be made to execute motions in a neighborhood of the vertical configuration. (Cf. [5], p. 408.) A similar analysis can be carried out for other values of the parameter α. We refer the reader to [15] for further details.

5. The Geometry of Averaged Mechanical Systems

If we assume the input function $v(\cdot)$ discussed in the preceding section has the form (24), then we see that the averaged potential V_A is invariant under a scaling of frequency $v(t) \to v(\lambda t)$ for $\lambda \neq 0$. Hence the form of V_A contains only partial information regarding the stability of the system, since the hypothesis of Theorem 3 requires both that λ is large enough so that the critical point of V_A is a strict local minimum, <u>and</u> that λ is large enough so that classical averaging theory applies. Nevertheless, further analysis may proceed based on knowledge of V_A alone. Based on the conclusions of Theorem 3, we might conjecture that if the frequency of the forcing is sufficiently high and if q_0 is a strict local minimum of V_A, then

$$\frac{d}{dt}\frac{\partial\hat{L}}{\partial\dot{q}} - \frac{\partial\hat{L}}{\partial q} = 0 \tag{35}$$

will execute motions in a neighborhood of $(q, \dot{q}) = (q_0, 0)$. An essential assumption in proving Theorem 3 was that dissipation was present, and we are now interested in investigating circumstances under which the result also holds for the conservative system. The required analysis must be carried out along different lines since there is no way to take advantage of asymptotic stability of a

linearization. We shall show, for a certain prototypical system, that in the absence of dissipation, stability will be the effect of a nonholonomic response of the system to the periodic forcing. To pursue our analysis, we linearize (35) about the point $(q_0, 0)$: Substituting $q = q_0 + \delta q$ into (35) and retaining terms up to first order in δq, we obtain

$$M(q_0)\ddot{\delta q} + \frac{\partial V_a}{\partial q}(q_0, v) + \frac{\partial^2 V_a}{\partial q^2}(q_0, v)\delta q$$

$$+ \Sigma_{j=1}^m \dot{v}_j(A_j(q_0) + \frac{\partial A_j}{\partial q}(q_0)\delta q) + \Sigma_{j=1}^m v_j(\frac{\partial A_j}{\partial q} - \frac{\partial A_j}{\partial q}^T)\dot{\delta q} = 0. \tag{36}$$

Two features of this system are of interest: (i) It is jointly bilinear in the inputs v, $\dot{v}$ and states q, $\dot{q}$, and (ii) the linearization is about a point which may or may not be an equilibrium of the system. When the point q_0 is not an equilibrium of the forced system, V. Solo ([16]) has suggested the term "hovering motion" replace the term "stable motion." We shall pursue the analysis of both types of motion under the general heading of "stability." The advantage of studying the system (36) is that stability analysis may be pursued using powerful linear tools such as Floquet theory. This may be illustrated in terms of the cart-pendulum problem.

Example 8 (Example 7 reprise) Recall (34)

$$\hat{L}(\theta, \dot{\theta}; v) = \frac{1}{2}\ell\dot{\theta}^2 + v\dot{\theta}\cos(\theta - \alpha) + g\cos\theta.$$

Specializing the terms in (36), we have $M(\theta) = \ell$, $A(\theta) = \cos(\theta - \alpha)$, and $V_a(\theta) = -g\cos\theta$. Hence, we may write the variational equation corresponding to (33) with $\dot{v} = \ddot{r}$ as

$$\frac{d}{dt}\begin{pmatrix}\delta\theta \\ \dot{\delta\theta}\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -\ell^{-1}g\cos\theta_0 + \frac{\dot{v}}{\ell}\sin(\theta_0 - \alpha) & 0\end{pmatrix}\begin{pmatrix}\delta\theta \\ \dot{\delta\theta}\end{pmatrix} + \begin{pmatrix}0 \\ -\frac{g}{\ell}\sin\theta_0 - \frac{\dot{v}}{\ell}\cos(\theta_0 - \alpha)\end{pmatrix}. \tag{37}$$

Pursuing an analysis of the second case considered in Example 6, we set $\alpha = \frac{\pi}{2}$ and consider the critical point $\theta_0 = \pi$ of the averaged potential. The above equations specialize to

$$\frac{d}{dt}\begin{pmatrix}\delta\theta \\ \dot{\delta\theta}\end{pmatrix} = \begin{pmatrix}0 & 1 \\ \frac{g+\dot{v}}{\ell} & 0\end{pmatrix}\begin{pmatrix}\delta\theta \\ \dot{\delta\theta}\end{pmatrix}. \tag{38}$$

In general, if we have a linear differential equation with periodic coefficients of the form

$$\dot{x}(t) = F(t)x(t), \tag{39}$$

where the coefficient matrix has entries which are bounded, piecewise analytic, and periodic of period $T > 0$, then Floquet theory implies that the trivial solution to (39) will be (Lyapunov) stable if all eigenvalues of the fundamental matrix $\Phi(T, 0)$ are distinct and lie on the unit circle. When this occurs, we have the following

Proposition 4 *Let $\Phi(t, 0)$ denote the fundamental matrix solution to (39), and suppose $\Phi(T, 0)$ ($T > 0$ is the fundamental period of v) has distinct eigenvalues lying on the unit circle. The solution to (39) sampled every T units of time evolves on an ellipsoid $x^T M x = 1$.*

Proof: If all eigenvalues of $\Phi(T,0)$ are distinct and lie on the unit circle, there is a nonsingular matrix P such that $\Lambda = P^{-1}\Phi(T,0)P$ is block diagonal with 0's and 2×2 blocks of the form $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ on the diagonal. Now $\Lambda\Lambda^T = I$, and from this it follows that $\Phi(T,0)^T M \Phi(T,0) = M$ where $M^{-1} = PP^T$. With the appropriate scaling of M, the proposition follows. □

Suppose we introduce the phase-shift $v(t) \mapsto v(t+\epsilon)$. Then it is easy to show that the time-T invariant ellipsoid is defined by the quadratic form $M_\epsilon \equiv \Phi(0,\epsilon)^T M \Phi(0,\epsilon)$. The union of the family of invariant ellipsoids, $\cup\{x^T M_\epsilon x = 1 : 0 \le \epsilon \le T\}$, provides an approximation of the region in the state-space through which the trajectories of the forced system (39) pass. Indeed, if $x(0) = x_0$ and $x_0^T M x_0 = 1$, then since $x(s)^T M_s x(s) = x_0^T M x_0$, the solution to (39) with initial conditions $x(0) = x_0$ lies in this union of ellipsoids.

A general theory of how well this set describes the set of points through which the solution passes does not presently exist. The cart-pendulum problem illustrates what is involved, however.

Example 9 (Example 8 reprise) Consider the linearized dynamics for the cart-pendulum problem (38). Letting $u(t) = -(g + \ddot{v}(t))/\ell$ and $y = (y_1, y_2) = (\delta\theta, \dot{\delta\theta})$, this represents the well-studied Hill's equation. (See e.g. [11].) We shall conclude this section with a discussion of the relationship between the geometry of solutions of (37) and the averaged potential introduced in the preceding section. Suppose the input u is a square wave

$$u(t) = \begin{cases} -\gamma & \text{if } 0 \le t < h; \\ \beta & \text{if } h \le t < 2h; \\ u(t-2h) & \text{if } 2h \le t, \end{cases}$$

(β, $\gamma > 0$). The fundamental matrix $\Phi(t,0)$ has determinant 1 for all t, and at time $T = 2h$ may be explicitly written as

$$\Phi(2h,0) = \begin{pmatrix} \cos\sqrt{\beta}h & \frac{1}{\sqrt{\beta}}\sin\sqrt{\beta}h \\ -\sqrt{\beta}\sin\sqrt{\beta}h & \cos\sqrt{\beta}h \end{pmatrix} \begin{pmatrix} \cosh\sqrt{\gamma}h & \frac{1}{\sqrt{\gamma}}\sinh\sqrt{\gamma}h \\ \sqrt{\gamma}\sinh\sqrt{\gamma}h & \cosh\sqrt{\gamma}h \end{pmatrix}.$$

As our previous analysis suggests, the solution $(y_1, y_2) = (0,0)$ will be stable precisely when the eigenvalues of $\Phi(2h,0)$ are on the unit circle. The characteristic polynomial of $\Phi(2h,0)$ is

$$p(s) = s^2 - \mu(\beta,\gamma,h)s + 1$$

where $\mu(\beta,\gamma,h) = 2\cos\sqrt{\beta}h\cosh\sqrt{\gamma}h + (\sqrt{\frac{\gamma}{\beta}} - \sqrt{\frac{\beta}{\gamma}})\sin\sqrt{\beta}h\sinh\sqrt{\gamma}h$. The roots of this polynomial will be on the unit circle precisely when $|\mu(\beta,\gamma,h)| \le 2$. When $\beta = \gamma = 1$, for example, one sees that there is a sequence of roots of the equation $\mu(1,1,h) = 2$: $h_0 = 0$, $h_1 = 1.8751$, $h_2 = 4.73004, \ldots$ with $h_n \to \infty$ as $n \to \infty$. In the interval (h_{2k}, h_{2k+1}), the trivial solution to Hill's equation ((38) will be stable, while for $h \in (h_{2k+1}, h_{2k+2})$, the solution will be unstable.

Applying the analysis of the preceding section to this system, Theorem 3 implies that for all sufficiently small $h > 0$, the origin will be a stable equilibrium for (38). A more refined picture emerges from an analysis of the invariant ellipsoids.

Proposition 5 *Given a* 2×2 *matrix* $\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$ *with the property that* $\det \Phi = 1$, *there is a symmetric* 2×2 *solution* $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}$ *to the matrix equation*

$$\Phi^T M \Phi = M$$

given explicitly by

$$\begin{pmatrix} m_{11} \\ m_{12} \\ m_{22} \end{pmatrix} = \begin{pmatrix} -2\phi_{21} \\ \phi_{11} - \phi_{22} \\ 2\phi_{12} \end{pmatrix}.$$

This solution is unique to within a scale factor.

Proof: A direct computation, using the hypothesis $\det \Phi = 1$, verifies that the solution we have proposed does indeed satisfy the equation. To prove that this solution is unique up to a scale factor, it is only necessary to show that the linear mapping $M \mapsto \Phi^T M \Phi$ has rank 2. To do this, we shall prove that the images of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent. Suppose we have scalars a and b such that

$$a \begin{pmatrix} \phi_{11}^2 & \phi_{11}\phi_{12} \\ \phi_{11}\phi_{12} & \phi_{12}^2 \end{pmatrix} + b \begin{pmatrix} \phi_{21}^2 & \phi_{21}\phi_{22} \\ \phi_{21}\phi_{22} & \phi_{22}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is straightforward to show that this equation is equivalent to

$$\begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} \begin{pmatrix} a\phi_{11} \\ b\phi_{21} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} \begin{pmatrix} a\phi_{12} \\ b\phi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\det \Phi = 1$, this is equivalent to

$$\begin{pmatrix} a\phi_{11} \\ b\phi_{21} \end{pmatrix} = \begin{pmatrix} a\phi_{12} \\ b\phi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since we cannot have $\phi_{11} = \phi_{21} = 0$, we must have $a = 0$. Similarly, $b = 0$, proving the rank of the linear mapping is 2, as claimed. This proves the proposition. □

Remark 9 Given $\Phi(2h, 0)$ as in Example 9, let M be the corresponding invariant quadratic form specified by Proposition 5. The condition that M may, after a multiplication by a suitable scalar, be taken to be positive definite is that $m_{11}m_{22} - m_{12}^2 > 0$. Note that this is equivalent to the condition that $\Phi(2h, 0)$ has distinct eigenvalues on the unit circle: i.e. to $(\phi_{11} + \phi_{22})^2 - 4 < 0$. □

Given a square wave input $u(\cdot)$ as prescribed above, we write the corresponding solution to

$$\Phi(2h, 0)^T M \Phi(2h, 0) = M,$$

and normalize M with respect to the initial conditions such that $y(0)^T M y(0) = 1$. If M is positive definite, solutions $(y_1, y_2) = (\delta\theta, \dot{\delta\theta})$ of (38) which satisfy this equation evolve on the union of ellipses $\cup\{y^T M_\epsilon y = 1 : 0 \le \epsilon \le T\}$. A typical motion is shown in Figure 3, where the ellipses $y^T M_0 y = 1$ and $y^T M_h y = 1$ are also depicted.

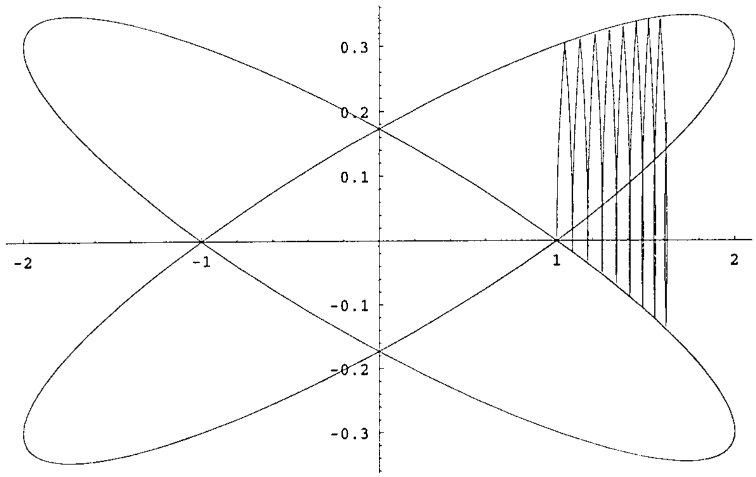

Figure 3: Phase portrait of solution to Hill's equation with two members of the family of invariant ellipsoids. Forcing is by square wave function $u(t)$ as described in the text with parameters $\alpha = \beta = 1$ and $h = 0.3$.

Our final objective will be to return to the question of how our results vary as the the frequency of the forcing is changed. This will provide an important complement to the discussion in the previous section where it was noted that the averaged potential V_A is invariant under a change in the frequency of the forcing. Returning to (38), it will be useful to now assume $\dot{v}(t)$ (instead of $u(t)$) is the square wave

$$\dot{v}(t) = \begin{cases} -\gamma & \text{if } 0 \le t < h; \\ \beta & \text{if } h \le t < 2h; \\ \dot{v}(t-2h) & \text{if } 2h \le t, \end{cases} \tag{40}$$

$(\beta,\, \gamma > 0)$. To study the effect of scaling the frequency, we define $v_\lambda(t) = v(\lambda t)$ so that

$$\dot{v}_\lambda(t) = \begin{cases} -\lambda\gamma & \text{if } 0 \le t < \frac{h}{\lambda}; \\ \lambda\beta & \text{if } \frac{h}{\lambda} \le t < \frac{2h}{\lambda}; \\ \dot{v}_\lambda(t - 2h/\lambda) & \text{if } t \ge \frac{2h}{\lambda}, \end{cases}.$$

From Proposition 2, we write the averaged potential for the cart-pendulum system (34) in the case $\alpha = \frac{\pi}{2}$ and the input has $\gamma = \beta$:

$$V_A(\theta) = -g\cos\theta + \frac{1}{2\ell}\sin^2\theta \cdot (\sigma^2 - \bar{v}^2).$$

A straightforward computation to evaluate $\bar{v}$, σ yields

$$V_A(\theta) = -g\cos\theta + \frac{\beta^2 h^2}{24\ell}\sin^2\theta. \tag{41}$$

$\theta = \pi$ is a critical point of this function for all values of the parameters, and for $g\ell < \frac{\beta^2 h^2}{12}$, it is a local minimum. This may be compared with the analysis of Example 7, and it is re-emphasized that the result is independent of the frequency parameter λ.

On the other hand, we may explicitly write the fundamental matrix solution $\Phi_\lambda(t,0)$ to (38) corresponding to the input v_λ. Assuming again that $\gamma = \beta$, we evaluate this at the fundamental period

$t = \frac{2h}{\lambda}$:

$$\Phi_\lambda(\frac{2h}{\lambda},0) = \begin{pmatrix} 1 - \frac{\beta h^2}{\ell\lambda} & \frac{2h}{\lambda} \\ \frac{(6g\ell - 2\beta^2 h^2)h}{3\lambda\ell^2} & 1 + \frac{\beta h^2}{\ell\lambda} \end{pmatrix} + o(\frac{1}{\lambda}).$$

According to Proposition 5, there is a corresponding invariant quadratic form specified by the symmetric 2×2 matrix

$$M = \begin{pmatrix} \frac{4h}{3\ell^2}(\beta^2 h^2 - 3g\ell) & -\frac{2\beta h^2}{\ell} \\ -\frac{2\beta h^2}{\ell} & 4h \end{pmatrix} \frac{1}{\lambda} + o(\frac{1}{\lambda}).$$

It is easy to check that this is positive definite for all sufficiently large λ precisely when $\beta^2 h^2 > 12g\ell$. We summarize this observation in the following.

Proposition 6 *Consider the cart-pendulum system (34) with inclination parameter value* $\alpha = \frac{\pi}{2}$ *and periodic forcing* $v(\cdot)$ *defined as in (40) with* $\gamma = \beta$. *Suppose the dynamics are linearized about the equilibrium* $\theta = \pi$ *of the averaged system to yield (38). Then the origin is stable (in the sense of Lyapunov) for (38) under all forcing* $v_\lambda(t) = v(\lambda t)$ *with* λ *sufficiently large if and only if* $\theta = \pi$ *is a strict local minimum of the averaged potential (41).* □

Remark 10 One might carry out a similar stability analysis with the assumption $\gamma = \beta$ relaxed. Any persistent bias of this type in the forcing, however, seems to destroy the stability we have observed. Thus, the shape of the wave-form forcing the system seems to be crucial. It is worth pointing out that numerical experiments indicate that the conclusions of Proposition 6 remain valid for any sufficiently regular periodic input of the form (24) (e.g. $v_\lambda(t) = \sin \lambda t$).

Remark 11 Proposition 6 represents a remarkable confluence of an analytical and geometrical analysis of stability. The fact that local minima of the averaged potential can be shown to be Lyapunov stable for motions of the forced system serves to further inspire the efforts we are pursuing to simply characterize the nonlinear response of mechanical systems to regular forms of forcing. It is worth noting that even for cases where $\alpha \neq \frac{\pi}{2}$ and θ_0 is a local minimum of the averaged potential V_A which does not correspond to an eqilibrium of the system (33), an extension of the argument used to prove Proposition 6 shows that the linearization (37) of (33) about θ_0 will execute bounded motions. A more complete treatment of stability of systems of the form (36) will appear elsewhere.

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3 Energy Methods for Stability of Bilinear Systems with Oscillatory Inputs

Energy Methods for Stability of Bilinear Systems with Oscillatory Inputs

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Key words: averaged potential, nonlinear control synthesis
open loop design

Abstract: A large body of recent literature has been devoted to the topic of motions of mechanical systems forced by oscillatory inputs. (See, e.g. [7], [12], [15], [17], and [29]!) A common feature in most of this work (with [7] being the exception) is that the results apply principally to kinematic problems, and the equations of motion do not involve drift terms. The main object of study in the present paper is the class of bilinear control systems with oscillatory (periodic) inputs. We shall show that included in this class are models of mechanical system dynamics which are obtained through a process of reduction and linearization about operating points. In particular, we study the stability of such systems under high frequency, periodic forcing. The *averaged potential* is defined for linearizations of the systems on the (reduced) configuration space, and it is shown that stable motions of the forced system are associated with minimum values of this quantity. We use both classical averaging theory as well as a novel geometric argument to provide parallel but independent assessments of the use of the *averaged potential* in carrying out a stability analysis. A salient feature of the geometric approach is that we are able to justify the use of an energy like quantity to determine Lyapunov stability in conservative mechanical systems. This makes contact with a growing body of literature on the use of energy methods for stability and control design (e.g. [7], [8], [18], [24], [26], [27]). We briefly describe the connection with previous work on the averaged potential.

1. INTRODUCTION

Although geometric nonlinear control theory has achieved a high degree of maturity over the past two decades, no substantial body of research on nonlinear control designs systematically utilizing the geometry of noncommuting vector fields had been reported until quite recently. (We cite Haynes and Hermes, [13], as a notable early effort along these lines, however.) Recently a number of researchers (e.g. [12], [15], [17], [20] [16], and [29]) have reported progress in the use of open-loop, oscillatory controls to steer certain classes of systems that are of interest in path-planning problems in robotics. The present paper is written in the same spirit, but our focus is on systems arising in dynamic rather than kinematic models.

In previously reported work ([4]-[7]), we studied stability of the dynamics of mechanical systems subject to oscillatory forcing. The overall goal of the research has been to understand system dynamics which are

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the net result of applying periodically varying forcing to a set of noncommuting vectorfields. One result has been our characterization of stable operating points as critical points of an energy-like quantity called the *averaged potential.* In particular, in [7], it was shown that critical points of the averaged potential which are also strict local minima are stable provided the frequency of the periodic forcing is sufficiently high. The results were established using classical averaging theory together with the adroit introduction of damping into the models. In the present paper, we show that the stability of motions in conservative dynamical models may be assessed in terms of the averaged potential with no assumption of dissipation in the system.

The paper is organized as follows. In Section 2, we study general bilinear systems forced by oscillatory inputs. It is shown that if periodic inputs are scaled by the same factor in frequency and amplitude, that given a certain assumption on average dissipation in the model, the long term behavior will be described by an averaged model involving mean and r.m.s. values of the input signal. This is a more-or-less direct consequence of the classical averaging theorem. In Section 3, a special class of mechanical systems models is considered. The models are obtained through a process of (Lagrangian) reduction and linearization about operating points. Such models, with oscillatory coefficients, have been treated extensively in the Russian literature. (See, for instance [11] and [30].) For the models which are of interest to us, it is shown that stable motions may be found in terms of a quantity we have called the *averaged potential.* Section 4 employs Floquet theory to prove that the averaged potential may be used to identify stable motions under oscillatory forcing even when dissipation is not present in the model.

The use of energy methods for analyzing the stability of mechanical systems has been pursued recently in a number of problem areas arising in the mechanics of rotating systems (for example, [3], [8], [18], [24], [26], and [27]). In [8] and [24], energy methods are employed to design control laws for the stabilization of rotating rigid bodies. While our idea of using the *averaged potential* for studying the stability of systems with inputs is fairly recent (See [4], [5], and [7] for details on the *averaged potential.*), there are antecedents of the concept in the analytical mechanics literature from the nineteenth century. Omohundro ([22]) describes efforts by Helmholz, Hertz, and others to develop a "kinetic theory of matter" wherein potential forces would arise from "hidden or forgotten degrees of freedom." Omohundro interprets forces attributable to such motions in terms of quantities he calls *pseudo-potentials.* Percival and Richards ([23]) briefly mention such quantities under the name *effective potentials.*

2. Bilinear Systems

Consider the bilinear control system

$$\dot{x} = (A + \sum_{i=1}^{m} u_i(t)B_i)x, \tag{1}$$

where $A, B_1, \ldots, B_m$ are constant $n \times n$ matrices, $x(t) \in \mathbf{R}^n$. For much of the analysis which follows, we shall only assume that the $u_i(\cdot)$'s are piecewise continuous functions defined on some interval $[0, t_f)$ where $0 < t_f \leq \infty$. This serves to distinguish our results from some of the published work on classical averaging where it is assumed that the forcing is continuous ([25]) or even C^2 ([14]). We shall also assume a strong nilpotency condition, namely that $B_iB_j = 0$ for all $i, j = 1, \ldots, m$. While this might appear to limit the

generality of our results, we shall find that the condition is always satisfied for a wide class of systems arising from models of mechanical system dynamics.

Certain elements of the control theory of such systems are well developed (see, e.g. [21]), but there are few general methods for design of control laws to achieve particular motion objectives. Exceptions to this include the optimal control laws studied in [2] and also the use of oscillatory inputs to approximate desired motions in [15] and [29]. While systems such as (1) are not amenable to treatment using the methods of [15] and [29] due to the presence of the "drift" term A, these results nevertheless suggest studying the effect of applying oscillatory inputs to (1).

Assume $\tilde{u}_1(\cdot), \ldots, \tilde{u}_m(\cdot)$ are periodic functions with (common) fundamental period $T > 0$. In order to apply classical averaging theory to study the motion of (1) we consider the effect of increasing the frequency of the forcing. Specifically, we study the dynamics of

$$\dot{x}(t) = (A + \sum \tilde{u}_i(\omega t) B_i) x(t)$$

as ω becomes large. The analysis proceeds by scaling time and considering $\tau = \omega t$. Let $z(\tau) = x(t)$. This satisfies the differential equation

$$\frac{dz}{d\tau} = \frac{1}{\omega}(A + \sum \tilde{u}_i(\tau) B_i) z. \tag{2}$$

Assuming $\omega > 0$ is large, and writing $\epsilon = \frac{1}{\omega}$, we see that (2) is in a form to which classical averaging theory applies.

Proposition 1 *Consider the bilinear system (1) with $u_i(\cdot) = \tilde{u}_i(\cdot)$ where for $i = 1, \ldots, m$, $\tilde{u}_i(\cdot)$ is continuous on $0 \le t < t_f \le \infty$ and periodic of period $T \ll t_f$. Let*

$$\bar{u}_i = \frac{1}{T} \int_0^T \tilde{u}_i(t)\, dt,$$

and let $y(\tau)$ be a solution of the constant coefficient linear system

$$\dot{y}(\tau) = \epsilon (A + \sum \bar{u}_i B_i) y(\tau). \tag{3}$$

If z_0 and y_0 are respective initial conditions associated with (2) and (3) such that $|z_0 - y_0| = \mathcal{O}(\epsilon)$, then $|z(\tau) - y(\tau)| = \mathcal{O}(\epsilon)$ on a time scale $\tau \sim \frac{1}{\epsilon}$.

This proposition is a straightforward application of classical averaging theory; see [25]. Unfortunately, it doesn't capture the full range of behaviors that are possible with different scalings of the magnitudes of the inputs $\tilde{u}_i$. For instance, consider the following second order system

$$\ddot{x}(t) + (\alpha + \beta u(t)) x(t) = 0. \tag{4}$$

When written in first order form, this is a special case of systems of the form (1). If we simultaneously scale the frequency and magnitude, $u(t) \mapsto \omega u(\omega t)$, the first order form of (4) becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha - \beta \omega u(\omega t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5}$$

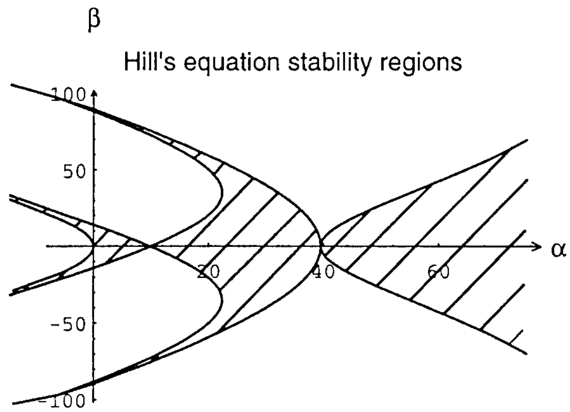

Figure 1: Stability of Hill's equation (4) with square-wave forcing (7).

and direct appeal to Proposition 1 is not possible. Nevertheless, re-writing the second order system in terms of the "slower" time scale $\tau = \omega t$ and $z(\tau) = x(t)$, we obtain

$$\frac{d^2z}{d\tau^2} + \Big(\big(\frac{\alpha}{\omega^2} + \frac{\beta}{\omega}u(\tau)\Big)z = 0. \tag{6}$$

When $u(\tau) = \cos\tau$, (6) is a standard rendering of Mathieu's equation (See, e.g. [28].), and in this case, the classical theory states that the origin is Lyapunov stable for sufficiently large ω provided $\alpha > -\frac{\beta^2}{2}$. This result will appear as a special case of the theory to be developed in the remainder of the paper.

Remark 1 The stability of (4) under periodic forcing has been most widely studied for the specific case $u(t) = \cos t$. There is growing interest in other types of inputs. (See, e.g., [19].) For square wave input

$$u(t) = \begin{cases} -1, & \text{if } 0 \le t < \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \le t < 1, \end{cases} \tag{7}$$

the regions in the (α, β) parameter space for which the origin $(x, \dot{x}) = (0, 0)$ is a stable rest point for (4) are depicted in Figure 1. Qualitatively, this picture is quite similar to the classical rendering of the stability regions corresponding to the forcing $u(t) = \cos t$. The boundary of the left-most stability region is approximately parabolic near the origin and satisfies $\alpha = -\frac{1}{48}\beta^2 + o(\beta^2)$. (Cf. [28], pp. 208-213.)

A particularly simple approach to averaging systems with high-frequency and high-amplitude inputs involves making a time-varying change of coordinates which replaces the inputs $u_i(\cdot)$ with their integrals. The coordinate transformation will turn out to have an especially simple form in the case that $B_iB_j = 0$ for all pairs $i, j = 1, \ldots, m$. Moreover, this condition will always be satisfied by the mechanical systems to be considered in the next section. We shall now further suppose that each $u_i(\cdot)$ is a zero-mean periodic function of period T. This assumption involves no loss of generality. If we wish to study a system (1) in which the mean of the inputs $u_i(\cdot)$ is not zero, we may write $u_i(t) = \bar{u}_i + \nu_i(t)$, where $\bar{u}_i = \frac{1}{T}\int_0^T u_i(s)\,ds$. The dynamics of (1) will be the same as for the system $\dot{x} = (\tilde{A} + \sum \nu_i B_i)x$, where $\tilde{A} = A + \sum \bar{u}_i B_i$ and which satisfies our assumption on the mean of the inputs. Letting

$$v_i(t) = \int_0^t u_i(s)\,ds,$$

it is easy to see that each $v_i(\cdot)$ is also a periodic function of period T. Given that $x(\cdot)$ satisfies (1), we transform to new coordinates $q(\cdot)$ using the formula

$$q(t) = (I - \sum v_i(t)B_i)x(t). \tag{8}$$

A straightforward calculation shows that

$$\dot{q}(t) = \Big(A + \sum v_i(t)[A, B_i] - \sum v_i(t)v_j(t)B_iAB_j\Big)q(t), \tag{9}$$

where $[A, B_i]$ denotes the usual matrix Lie bracket $AB_i - B_iA$.

The utility of changing to q-coordinates follows from the fact that in the slow time scale $\tau = \omega t$, (9) assumes a form to which classical averaging theory applies, provided the scaling factor ω is taken to be sufficiently large.

Theorem 1 *Consider the dynamical system*

$$\dot{x} = \Big(A + \sum_{i=1}^{m} \omega u_i(\omega t)B_i\Big)x, \tag{10}$$

where for $i = 1, \ldots, m$ $u_i(\cdot)$ is a piecewise continuous periodic function with period T and mean 0, and for $i, j = 1, \ldots, m$, $B_iB_j = 0$. Let $\epsilon = \frac{1}{\omega}$. Define for each $i = 1, \ldots, m$, the periodic function $v_i(t) = \int_0^t u_i(s)\,ds$, and let

$$\bar{v}_i = \frac{1}{T}\int_0^T v_i(s)\,ds, \qquad i = 1, \ldots, m$$

$$\sigma_{ij} = \frac{1}{T}\int_0^T v_i(s)v_j(s)\,ds, \qquad i, j = 1, \ldots, m.$$

Let $y(t)$ be a solution of the constant coefficient linear system

$$\dot{y} = \Big(A + \sum_{i,j}(\bar{v}_i\bar{v}_j - \sigma_{ij})B_iAB_j\Big)y. \tag{11}$$

Suppose the eigenvalues of (11) have negative real parts. Then there is a $t_1 > 0$ such that for $\omega > 0$ sufficiently large (i.e. for $\epsilon > 0$ sufficiently small), if x_0 and y_0 are respective initial conditions for (10) and (11) such that $|x_0 - y_0| = \mathcal{O}(\epsilon)$, then $|x(t) - y(t)| = \mathcal{O}(\epsilon)$ for all $t > t_1$.

Proof: Given the system (10) and the functions $v_i(\cdot)$ defined in the statement of the theorem, define the new state coordinates

$$q(t) = (I - \sum_{i=1}^{m} v_i(\omega t)B_i)x(t)$$

in terms of which the dynamics are expressed by (9). Changing to time variable $\tau = \omega t$ and state variable $z(\tau) = q(t)$, we have dynamics

$$\frac{dz}{d\tau} = \epsilon\Big(A + \sum v_i(\tau)[A, B_i] - \sum v_i(\tau)v_j(\tau)B_iAB_j\Big)z. \tag{12}$$

It is this system to which the averaging theorem ([25], Theorem 4.2.1, p. 71) applies, with the corresponding averaged dynamics given by

$$\frac{d\bar{z}}{d\tau} = \epsilon\Big(A + \sum \bar{v}_i[A, B_i] - \sum \sigma_{ij} B_i A B_j\Big)\bar{z}. \tag{13}$$

Specifically, if z_0, $\bar{z}_0$ are respective initial conditions of (12) and (13) such that $|z_0 - \bar{z}_0| = \mathcal{O}(\epsilon)$, then $|z(\tau) - \bar{z}(\tau)| = \mathcal{O}(\epsilon)$, on a time scale $\tau \sim \frac{1}{\epsilon}$. If the spectrum of (13) is in the left half-plane, then the conclusion is valid on the time interval $[0, \infty)$.

Reverting to the original time-scale, and writing $\bar{q}(t) = z(\omega t)$, we find that $\bar{q}$ satisfies

$$\dot{\bar{q}} = \Big(A + \sum \bar{v}_i[A, B_i] - \sum \sigma_{ij} B_i A B_j\Big)\bar{q}(t),$$

and given $|q(0) - \bar{q}(0)| = \mathcal{O}(\epsilon)$, it follows that $|q(t) - \bar{q}(t)| = \mathcal{O}(\epsilon)$, and provided the spectrum is in the left half-plane, this is valid on the time scale $[0, \infty)$.

Let $\bar{x}(t) = (I + \sum \bar{v}_i B_i)\bar{q}(t)$. This satisfies the constant coefficient linear differential equation

$$\dot{\bar{x}} = \Big(A + \sum_{i,j}(\bar{v}_i\bar{v}_j - \sigma_{ij})B_i A B_j\Big)\bar{x}.$$

Noting also that $x(t) = (I + \sum v_i(t)B_i)q(t)$, we write

$$\begin{aligned}
|x(t) - \bar{x}(t)| &= |(I + \textstyle\sum v_i B_i)q(t) - (I + \sum \bar{v}_i B_i)\bar{q}(t)| \\
&\le |q(t) - \bar{q}(t)| + |\textstyle\sum v_i B_i q(t) - \sum v_i B_i \bar{q}(t) + \sum (v_i - \bar{v}_i) B_i \bar{q}(t)| \\
&\le |q(t) - \bar{q}(t)| + |\textstyle\sum v_i B_i (q(t) - \bar{q}(t))| + \sum |v_i - \bar{v}_i||B_i \bar{q}(t)| \\
&\le \mathcal{M}_1|q(t) - \bar{q}(t)| + \mathcal{M}_2|\bar{q}(t)|,
\end{aligned}$$

where $\mathcal{M}_1 = \sup_{0\le t\le T}(1 + |v_i(t)B_i|)$ and $\mathcal{M}_2 = \sup_{0\le t\le T}(\sum |v_i(t) - \bar{v}_i||B_i|)$. The norms of vector quantities in this case are the ordinary finite dimensional Euclidean norms, and the matrix norms are the induced norms. Under the hypothesis that the spectrum of (11) lies in the left half-plane, it is also the case that the spectrum of (13) lies in the left half-plane. Hence, from the above, $|q(t) - \bar{q}(t)| = \mathcal{O}(\epsilon)$, for $\epsilon > 0$ sufficiently small. Moreover, if the eigenvalues of (13) are in the left half-plane, it follows that the equilibrium point $\bar{q} = 0$ is asymptotically stable, and $|\bar{q}(t)| \to 0$ as t increases. This proves the theorem. □

Remark 2 Although Theorem 1 is of some interest in characterizing the response of bilinear systems to periodic forcing, a limitation may be uncovered by re-examining (5). Again suppose $u(t) = \cos t$. Then $T = 2\pi$ and $v(t) = \sin t$. Hence $\bar{v} = 0, \sigma = 1/2$, and the coefficient matrix appearing in (11) is $\begin{pmatrix} 0 & 1 \\ -\alpha - \frac{1}{2}\beta^2 & 0 \end{pmatrix}$. Since the eigenvalues are purely imaginary, Theorem 1 does not apply. On the other hand, as remarked earlier, the origin is stable in the sense of Lyapunov for all sufficiently large values of ω.

3. Linear Lagrangian Systems

Consider a Lagrangian control system prescribed by a Lagrangian

$$L(x,\dot{x};v) = \frac{1}{2}\dot{x}^T M\dot{x} + \dot{x}^T Sx - \frac{1}{2}x^T Kx + \sum_{i=1}^{m} v_i \dot{x}^T C_i x$$

where M, K, and C_i are $n\times n$ symmetric matrices with M and K positive definite, and S is an $n\times n$ skew-symmetric matrix. (See [21] and [7] for details on Lagrangian control systems.) The corresponding dynamics are

$$M\ddot{x} + 2S\dot{x} + Kx + \sum u_i(t)C_i x = 0 \tag{14}$$

where $u_i(t) = \dot{v}_i(t)$ for $i = 1,\ldots,m$.

Remark 3 *Linear Lagrangian systems arising from a class of super-articulated mechanical system models.* Linear Lagrangian control systems of the form (14) arise as partially linearized models of so-called *super-articulated* mechanical systems. (See [4]-[7] for an introduction to super-articulated mechanical systems.) The starting point for a study is a Lagrangian $L(q,\dot{q})$ defined on the tangent bundle TQ of the configuration space of a mechanical system. Suppose that the generalized coordinates can be partitioned as $q = (q_1, q_2)$ and that exogenous generalized forces (control inputs) can be applied to only the coordinates q_2, while the coordinate variables comprising q_1 evolve freely. Specifically, we assume the equations of motion for the system take the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0, \tag{15}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = u, \tag{16}$$

where u is a p-vector of controls, and we suppose the mapping $\dot{q}_2 \mapsto \frac{\partial L}{\partial \dot{q}_2}$ is invertible. For the purposes of the model, we assume that the components $u_i(\cdot)$ are each piecewise analytic functions on $[0,\infty)$. For a wide variety of interesting physical systems, the terms on the left hand side of (15) depend explicitly on the generalized velocity $\dot{q}_2$ but not on the generalized coordinate q_2 itself. It is to such models that we direct our attention.

The problem we wish to consider is that of specifying $u(\cdot)$ to steer the state $(q_1, \dot{q}_1)$ in some desired fashion. While it is natural to view the velocities $\dot{q}_2$ as intermediate variables which transmit the effects of the controls to the states via the coupled equations (15)-(16), it will more be useful in what follows to view the variables $\dot{q}_2$ themselves as inputs, with the states $(q_1, \dot{q}_1)$ being determined by (15) and the required generalized forces $u(\cdot)$ determined by (16).

To treat a fairly general class of systems which will lead to models of the prescribed type, we further partition $\dot{q}_2 = (\dot{q}_{21}, \dot{q}_{22})$, and assume that the velocities $\dot{q}_{21}$ are held to certain constant values while $\dot{q}_{22} = v(t)$ is assumed to be a vector of inputs, trajectories of which may be chosen to control the system. By holding certain velocities $(\dot{q}_{21})$ constant, it will be seen that *gyroscopic* coupling is introduced among the components of the states $(q_1, \dot{q}_1)$. Physical examples of such coupling may be found in multibody system in which certain component bodies are driven to turn at a constant angular velocity relative to some body coordinate system. The mechanics of systems with gyroscopic coupling have been studied by

a number of researchers. See, for example Bloch, *et al.* [9]. For simplicity of exposition, we shall assume $\dot{q}_{21} = \vec{1} = (1, \ldots, 1)^T$.

Suppose the Lagrangian has the form $L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$ where $M(q)$ may be partitioned conformably with $(\dot{q}_1^T, \dot{q}_{21}^T, \dot{q}_{22}^T)$ as

$$M(q) = \begin{pmatrix} M_{11}(q_1) & M_{12}(q_1) & M_{13}(q_1) \\ M_{12}^T(q_1) & M_{22}(q_1) & M_{23} \\ M_{13}^T(q_1) & M_{23}^T & M_{33} \end{pmatrix}.$$

We assume q_1 is n-dimensional, as is x in (14), q_{21} is ν-dimensional for some $\nu \geq 0$, and q_{22} is m-dimensional, the dimension of the control input space in (14). Then (15) may be rendered

$$\frac{d}{dt}\frac{\partial L_R}{\partial \dot{q}_1} - \frac{\partial L_R}{\partial q_1} + (J_0(q_1) - J_0(q_1)^T)\dot{q}_1 + \sum_k [\dot{v}_k(t) M_{13}^k(q_1) + v_k(t)(J_k(q_1) - J_k(q_1)^T)\dot{q}_1] = 0 \tag{17}$$

where L_R is the *reduced Lagrangian*

$$L_R(q_1, \dot{q}_1) = \frac{1}{2}\dot{q}_1^T M_{11}(q_1)\dot{q}_1 - V_R(q_1)$$

with $V_R(q_1) = V(q_1) - \frac{1}{2}\vec{1}^T M_{22}(q_1)\vec{1}$. $J_0(q_1)$ is the $n \times n$ matrix whose (i,j)-th entry is the partial derivative with respect to q_j of the sum of entries in the i-th row of the sub-block $M_{12}(q_1)$, $M_{13}^k(q_1)$ is the k-th column of the sub-block $M_{13}(q_1)$, and $J_k(q_1)$ is the Jacobian matrix $\frac{\partial M_{13}^k}{\partial q_1}\big|_{q_1}$. Suppose $q_{10} \in \mathbf{R}^n$ and $(q_1, \dot{q}_1) = (q_{10}, 0)$ is a rest point of (15) for all values of $\dot{q}_2$. If we expand the dynamics (17) about the rest point and retain terms which are jointly first order in $q_1, \dot{q}_1$, we obtain

$$M\ddot{q} + S\dot{q} + Kq + \sum [\dot{v}_k(t) C_k q + v_k (C_k - C_k^T)\dot{q}] = 0, \tag{18}$$

where $q = \delta q_1$ is the variational term, $M = M_{11}(q_{10})$, $K = \frac{\partial^2 V_R}{\partial q_1^2}(q_{10})$, $S = J_0(q_{10}) - J_0^T(q_{10})$, and $C_k = \frac{\partial M_{13}^k}{\partial q_1}(q_{10})$.

Note that in (18), the matrices C_k are not necessarily symmetric. The presence of terms $C_k - C_k^T \neq 0$ indicates a form of "inertial coupling" between the inputs v and the states which is not present in (14), and systems of the form (18) thus appear to be more general and to include (14) as a special case. Indeed, the case in which the above linearization leads to symmetric coefficient matrices C_i is somewhat special, although it includes a number of interesting sub-cases including systems in which dim $q_1 = 1$. There appear to be some fundamental differences between the cases of symmetric and non-symmetric C_i's. While a parallel theory of systems with non-symmetric C_i's as described in this section (where the models include the effects of dissipation) could be described, we shall not do this. The results of the following section do not generalize as easily, and we shall conclude the paper with a remark detailing the differences between the two cases. □

Example 1 A well known physical phenomenon which may be used to illustrate our theory is stabilization of the inverted pendulum by means of oscillatory forcing of the point of suspension. (Cf. Arnold *et al.*, [1], pp. 152-153.) The equation for this system is

$$\ddot{\theta} + (\frac{g + \ddot{y}}{\ell})\sin\theta = 0$$

where ℓ is the length, g is the gravitational acceleration, and $y(t)$ is the vertical height of the point of suspension at time t. While y could be viewed as a physical coordinate, we shall instead view $\ddot{y}$ as a control input, which we rename u. $(\theta,\dot{\theta}) = (\pi, 0)$ is a rest point for the system for all choices of u, and carrying out the above linearization, (14) specializes to

$$\ddot{\theta} - (\frac{g+\ddot{y}}{\ell})\theta = 0. \tag{19}$$

□

To make contact with the preceding section, we rewrite equation (14) in first-order form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -2M^{-1}S \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sum_{i=1}^{m} u_i(t) \begin{pmatrix} 0 & 0 \\ -M^{-1}C_i & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{20}$$

where $x_1 = x$ and $x_2 = \dot{x}$. One would expect that systems of this form do not meet the hypotheses of Theorem 1, since such models apply to *conservative* mechanical systems. It is fairly standard, however, to introduce dissipation to such models by means of a quadratic (Rayleigh) dissipation function $\frac{1}{2}\dot{x}^T D\dot{x}$, where D is a positive definite symmetric matrix. With such dissipation included (20) is rewritten as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}(2S+D) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sum_{i=1}^{m} u_i(t) \begin{pmatrix} 0 & 0 \\ -M^{-1}C_i & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{21}$$

For $i = 1,\ldots,m$, let $w_i(\cdot)$ be a piecewise continuous periodic function of period T, having zero mean, and consider the dynamical system (20) (resp. (21)) forced by $u_i(t) = \omega w_i(\omega t)$ for some $\omega > 0$. Both (20) and (21) have the form (10). The averaged system corresponding to (21) takes the form

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K + \sum_{i,j}(\bar{v}_i\bar{v}_j - \sigma_{ij})M^{-1}C_iM^{-1}C_j & -M^{-1}(2S+D) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{22}$$

(Cf. (11).) Note that this system is associated with the constant coefficient Lagrangian

$$\bar{L}(y,\dot{y}) = \frac{1}{2}\dot{y}^T M\dot{y} - \frac{1}{2}x^T\Big(K + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)C_iM^{-1}C_j\Big)y + \dot{y}^T Sy,$$

with Rayleigh dissipation function $\frac{1}{2}\dot{y}^T D\dot{y}$. The quantity

$$\frac{1}{2}y^T\Big(K + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)C_iM^{-1}C_j\Big)y$$

plays the role of a potential and we are lead to the following.

Definition 1 *For the Lagrangian system (21) with periodic inputs $u_i(t) = \omega w_i(\omega t)$ as prescribed above and given the related averaged quantities $\bar{v}_i$ $(i = 1,\ldots,m)$ and σ_{ij} $(i,j = 1,\ldots,m)$ as defined in Theorem 1, we define the* averaged potential

$$V_A(x) = \frac{1}{2}x^T\Big(K + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)C_iM^{-1}C_j\Big)x.$$

Remark 4 It is easy to see that this quantity coincides with the *averaged potential* defined previously for mechanical systems in [7]

As in [7], the averaged potential may be used to determine the stability characteristics of the dynamical system (21) under the influence of oscillatory forcing. The following theorem gives the result more precisely.

Theorem 2 *Let $w_i(\cdot)$ be a piecewise continuous periodic function of period T and mean 0 for $i = 1, \ldots, m$. Define for each $i = 1, \ldots, m$, the periodic function $v_i(t) = \int_0^t w_i(s)\, ds$, and let*

$$\bar{v}_i = \frac{1}{T}\int_0^T v_i(s)\, ds, \qquad i = 1, \ldots, m$$

$$\sigma_{ij} = \frac{1}{T}\int_0^T v_i(s)v_j(s)\, ds, \qquad i, j = 1, \ldots, m.$$

Suppose D is positive definite, and let the system (21) be forced by the oscillatory inputs $u_i(t) = \omega w_i(\omega t)$ for $i = 1, \ldots, m$. If the averaged potential

$$V_A(x) = \frac{1}{2}x^T\left(K + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)C_iM^{-1}C_j\right)x$$

is a positive definite quadratic form, the state $(x_1, x_2) = (0, 0)$ is (asymptotically) stable for all sufficiently large ω.

Proof: Suppose the averaged potential is positive definite. Then the "energy"

$$E = \frac{1}{2}y_2^TMy_2 + \frac{1}{2}y_1^T\left(K + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)C_iM^{-1}C_j\right)y_1$$

is a positive definite quadratic form on (y_1, y_2)-space. Differentiating this along trajectories of (22), we obtain

$$\frac{d}{dt}E = -y_2^TDy_2.$$

Thus E serves as a Lyapunov function, and invoking LaSalle's invariance principle, all trajectories of (22) approach the set of points where $y_2 = 0$. Since the only solution to (22) with $y_2 \equiv 0$ is $(y_1, y_2) = (0, 0)$, we conclude that the origin is an asymptotically stable equilibrium. Hence the eigenvalues of (22) must lie in the left half-plane, and the conclusion of the theorem follows from Theorem 1. □

Theorem 2 relies on classical averaging theory to establish the connection between the averaged potential and stability of forced systems. It will be shown in the next section that the result is geometric in nature, and the stability analysis may be applied to conservative systems as well.

4. A Geometric Theory of Averaging

As pointed out in [7], the *averaged potential* provides useful information about the stability of motion even in the absence of dissipation. A natural tool in this study is Floquet theory, and while it may be inconvenient to analyze an explicit representation of the time-T mapping associated with a system of the form (10), the following lemma provides a useful approximation. The hypotheses adopted in this section are essentially the same as in the preceding two, but no assumptions are made regarding the asymptotic stability of the origin for the averaged system.

Lemma 1 *Consider the dynamical system*

$$\dot{x} = \Big(A + \sum_{i=1}^{m} \omega w_i(\omega t) B_i\Big) x,$$

where for $i = 1, \ldots, m$ $w_i(\tau)$ is a piecewise continuous periodic function with period T and mean 0 and continuous from the right for all $\tau \in [0, T]$. Suppose further that for $i, j = 1, \ldots, m$, $B_i B_j = 0$. Let $\epsilon = \frac{1}{\omega}$. Define for each $i = 1, \ldots, m$, the periodic function $v_i(t) = \int_0^t w_i(s)\, ds$. Then

$$\Phi_\epsilon(T, 0) = I + \int_0^T \exp\Big[-\sum v_i(\eta) B_i\Big] \cdot A \cdot \exp\Big[\sum v_i(\eta) B_i\Big]\, d\eta \cdot \epsilon + o(\epsilon)$$

is the transition matrix of

$$\frac{dy}{d\tau} = \Big(A\epsilon + \sum w_i(\tau) B_i\Big) y(\tau). \tag{23}$$

Proof: Let $\tau = \omega t$ and define $y(\tau) = x(t)$. If $\Phi_\epsilon(\tau, s)$ is the transition matrix of the system (23), then

$$\frac{\partial \Phi_\epsilon}{\partial \tau} = \Big(A\epsilon + \sum w_i(\tau) B_i\Big) \Phi_\epsilon(\tau, s),$$

with the derivative being taken from the right at points of discontinuity of any of the w_i's. Differentiating both sides with respect to ϵ, we find

$$\frac{\partial^2 \Phi_\epsilon}{\partial \tau \partial \epsilon} = A \Phi_\epsilon(\tau, s) + \Big(A\epsilon + \sum w_i(\tau) B_i\Big) \frac{\partial \Phi_\epsilon}{\partial \epsilon}(\tau, s).$$

It follows from the variation of constants formula that

$$\frac{\partial \Phi_\epsilon}{\partial \epsilon}(\tau, s) = \Phi_\epsilon(\tau, s) \frac{\partial \Phi_\epsilon}{\partial \epsilon}(s, s) + \int_s^\tau \Phi_\epsilon(\tau, \eta)\, A\, \Phi_\epsilon(\eta, s)\, d\eta.$$

Since $\frac{\partial \Phi_\epsilon}{\partial \epsilon}(s, s) = 0$, the first term on the right-hand side of this equation vanishes.

Letting $\epsilon = 0$, we see that $\Phi_0(\tau, s)$ is just the transition matrix associated with the coefficient matrix $\sum_{i=1}^m w_i(\tau) B_i$, and under the assumption $B_i B_j = 0$ for all i, j, we may write

$$\Phi_0(\tau, s) = \exp\Big[\int_s^\tau \sum w_i(\eta) B_i\, d\eta\Big].$$

Hence

$$\frac{\partial \Phi_\epsilon}{\partial \epsilon}\Big|_{\epsilon=0}(\tau,s) = \int_s^\tau \exp\Big[\int_\eta^\tau \sum w_i(\xi)B_i\,d\xi\Big]\cdot A\cdot \exp\Big[\int_s^\eta \sum w_i(\xi)B_i\,d\xi\Big]\,d\eta.$$

Note also, since we have assumed each $w_i(\cdot)$ to have zero mean, we have $\Phi_0(T,0)=I$. The formula for the time-T map may thus be given in terms of a series expansion with respect to ϵ:

$$\begin{aligned}\Phi_\epsilon(T,0) &= I + \int_0^T \exp\Big[\int_\eta^T \sum w_i(\xi)B_i\,d\xi\Big]\cdot A\cdot \exp\Big[\int_0^\eta \sum w_i(\xi)B_i\,d\xi\Big]\,d\eta\cdot\epsilon + o(\epsilon)\\ &= I + \int_0^T \exp\Big[-\sum v_i(\eta)B_i\Big]\cdot A\cdot \exp\Big[\sum v_i(\eta)B_i\Big]\,d\eta\cdot\epsilon + o(\epsilon).\end{aligned}$$

This proves the lemma. □

For linear Lagrangian systems without damping, the expansion derived in Lemma 1 corresponding to

$$\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} = \begin{pmatrix}0 & I\\ -M^{-1}K & -2M^{-1}S\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \sum_{i=1}^m \omega w_i(\omega t)\begin{pmatrix}0 & 0\\ -M^{-1}C_i & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{24}$$

takes the form

$$\Phi_\epsilon(T,0) = \begin{pmatrix} I - \sum \bar v_i M^{-1}C_i T\epsilon & IT\epsilon\\ -M^{-1}(K - 2\sum \bar v_i S M^{-1}C_i + \sum \sigma_{ij}C_i M^{-1}C_j)T\epsilon & I - 2M^{-1}S + \sum \bar v_i M^{-1}C_i T\epsilon\end{pmatrix} + o(\epsilon). \tag{25}$$

The stability analysis of (24) will be carried out in terms of iterates of the time-T transition matrix, $\Phi_\epsilon(T,0)$. More specifically, it follows from Floquet theory that the origin will be stable under the motion (24) when iterates of $\Phi_\epsilon(T,0)$ are stable. (See Guckenheimer and Holmes, [14], pp. 24-25, for a discussion of the relevant Floquet theory.)

Lemma 2 *Let $\Phi_\epsilon(T,0)$ be the transition matrix associated with (24) via equation (25), where T is the common fundamental period of the $w_i(\cdot)$'s. If $\Phi_\epsilon(T,0)$ has* distinct *eigenvalues lying on the unit circle, then there is a symmetric positive definite matrix $\mathcal{M}$ such that*

$$\Phi_\epsilon(T,0)^T\mathcal{M}\Phi_\epsilon(T,0) = \mathcal{M}. \tag{26}$$

Conversely, if $\Phi_\epsilon(T,0)$ leaves a symmetric positive definite $\mathcal{M}$ invariant as in (26), then $\Phi_\epsilon(T,0)$ has eigenvalues on the unit circle, and iterates $\Phi_\epsilon(T,0)^n$ remain bounded as $n\to\infty$.

Proof: If all eigenvalues of $\Phi(T,0)$ are distinct and lie on the unit circle, there is a nonsingular matrix P such that $\Lambda = P^{-1}\Phi(T,0)P$ is block diagonal with 0's and 2×2 blocks of the form $\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$ on the diagonal. Now $\Lambda\Lambda^T = I$, and from this it follows that $\Phi(T,0)^T\mathcal{M}\Phi(T,0) = \mathcal{M}$ where $\mathcal{M}^{-1} = PP^T$.

On the other hand, if $\Phi_\epsilon(T,0)$ leaves invariant a symmetric positive definite $\mathcal{M}$, let $\mathcal{M}^{1/2}$ be the unique positive definite symmetric square root of $\mathcal{M}$. It is easy to see that $\mathcal{M}^{1/2}\Phi_\epsilon(T,0)\mathcal{M}^{-1/2}$ is an orthogonal matrix. Hence $\mathcal{M}^{1/2}\Phi_\epsilon(T,0)\mathcal{M}^{-1/2}$ has eigenvalues on the unit circle, and the same conclusion for $\Phi_\epsilon(T,0)$ is an immediate consequence. □

The remaining results complete our stability theory by establishing conditions under which $\Phi_\epsilon(T,0)$ given in (25) leaves a symmetric positive definite quadratic form invariant.

Lemma 3 *For the system (24) evolving in $\mathbf{R}^{2n}$, let $\Phi_\epsilon(T,0)$ as given in (25) be partitioned into four $n\times n$ blocks*

$$\Phi_\epsilon = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}.$$

Suppose Φ_ϵ is symplectic, satisfying $\Phi_\epsilon^T J \Phi_\epsilon = J$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the standard symplectic bilinear form on $\mathbf{R}^{2n}$. Then Φ_ϵ also leaves a symmetric bilinear form $\mathcal{M}$ invariant. This form may be explicitly written

$$\mathcal{M} = \begin{pmatrix} -\Phi_{21} - \Phi_{21}^T & \Phi_{11}^T - \Phi_{22} \\ \Phi_{11} - \Phi_{22}^T & \Phi_{12} + \Phi_{12}^T \end{pmatrix}. \tag{27}$$

Proof: The proof is a straightforward verification that the linear (in $\mathcal{M}$) equation $\Phi^T \mathcal{M} \Phi = \mathcal{M}$ is satisfied. □

Theorem 3 *Consider (21) and the corresponding averaged system (22) (with $D = 0$). Assume that the inputs $u_i(t) = \omega w_i(\omega t)$ and the averaged quantities $\bar{v}_i$ and σ_{ij} are as in Theorem 2, and suppose further that $S = 0$. If the averaged potential is positive definite, the origin is stable in the sense of Lyapunov under the motion of (20) provided ω is sufficiently large.*

Proof: Change variables to $z_i = M^{\frac{1}{2}} x_i$ where $M^{\frac{1}{2}}$ is the unique symmetric, positive-definite square-root of the matrix M. The equation (20) becomes

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\tilde{K} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \sum_{i=1}^{m} u_i(t) \begin{pmatrix} 0 & 0 \\ -\tilde{C}_i & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{28}$$

where $\tilde{K} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$, $\tilde{C}_i = M^{-\frac{1}{2}} C_i M^{-\frac{1}{2}}$. Letting $u_i(t) = \omega w_i(\omega t)$, the time-$T$ map Φ_ϵ of Lemma 1 for this system takes the form

$$\Phi_\epsilon(T,0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} -\sum \bar{v}_i \tilde{C}_i & I \\ -(\tilde{K} + \sum \sigma_{ij} \tilde{C}_i \tilde{C}_j) & \sum \bar{v}_i \tilde{C}_i \end{pmatrix} \cdot T\epsilon + o(\epsilon).$$

We must find necessary and sufficient conditions under which it is possible to to solve $\Phi_\epsilon^T \mathcal{M} \Phi_\epsilon = \mathcal{M}$ for a symmetric, positive definite $\mathcal{M}$. Since all coefficient matrices appearing in (28) belong to the symplectic Lie algebra $sp(n, \mathbf{R})$, $\Phi_\epsilon = \Phi_\epsilon(T,0)$ belongs to the group of $2n \times 2n$ symplectic matrices, which satisfy $\Phi^T J \Phi = J$. Hence from Lemma 3, we conclude that $\Phi_\epsilon(T,0)$ also leaves the symmetric quadratic form (27) invariant. Using this explicit representation, we write

$$\mathcal{M}_\epsilon = \begin{pmatrix} \tilde{K} + \sum \sigma_{ij} \tilde{C}_i \tilde{C}_j & -\sum \bar{v}_i \tilde{C}_i \\ -\sum \bar{v}_i \tilde{C}_i & I \end{pmatrix} 2T\epsilon + o(\epsilon).$$

Now $\mathcal{M}_\epsilon$ will be positive definite for all sufficiently small ϵ (large ω) precisely when the the matrix

$$\begin{pmatrix} \tilde{K} + \sum \sigma_{ij} \tilde{C}_i \tilde{C}_j & -\sum \bar{v}_i \tilde{C}_i \\ -\sum \bar{v}_i \tilde{C}_i & I \end{pmatrix}$$

is positive definite. This is more easily analyzed in the coordinate system

$$\begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} = \begin{pmatrix} I & \sum \bar{v}_i \tilde{C}_i \\ 0 & I \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

with the transformation from z to $\hat{z}$ changing the expression of the matrix in question to

$$\begin{pmatrix} \tilde{K} + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)\tilde{C}_i\tilde{C}_j & 0 \\ 0 & I \end{pmatrix}.$$

The positive definiteness of this matrix is clearly equivalent to the positive definiteness of the sub-block $\tilde{K} + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)\tilde{C}_i\tilde{C}_j$. This matrix is obviously positive definite if and only if the x-coordinate averaged potential matrix $K + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)C_iM^{-1}C_j$ is positive definite. This proves the theorem. □

Remark 5 The assumption that $S = 0$ appears to be needed for the construction of the invariant quadratic form in terms of which we determine the stability of this system. Beyond this, however, it has been shown that the stability of linear systems with gyroscopic terms is ambiguous. In particular, it was shown in [6] and also [9] that in cases where K (or $K + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)C_iM^{-1}C_j$) is not positive definite, the system (14) will have right half plane eigenvalues if small amounts of dissipation are formally introduced, as was done in Section 3. Thus, the conditions of the theorem would stand no chance of being necessary and sufficient in the presence of non-zero S.

Example 2 (*Example 1, reprise.*) In Example 1, the linear Lagrangian system corresponding to the inverted pendulum with oscillatory support, was written (19):

$$\ddot{\theta} - (\frac{g + \ddot{y}}{\ell})\theta = 0.$$

Identifying the standard terms in (14) for this example we have $K = -g/\ell$, $M = 1$, and $C = -1/\ell$. For the oscillatory input $u(t) = \omega^2 \sin(\omega t)$, we have $\bar{v} = 0$ and $\sigma^2 = \omega^2/2$. The averaged potential for this system is $(-\frac{g}{2\ell} + \frac{\omega^2}{4\ell^2})\theta^2$. The criterion for stability is that the point $\theta = 0$ is a strict local minimum of this function, and this will obviously be the case precisely when $\omega^2 > 2g\ell$. This, of course, coincides with the classical stability result, (Cf. [1], p. 153.) for this problem. □

Example 3 Consider the electric circuit in Figure 2. The current in the circuit satisfies the second order differential equation

$$L\frac{d^2}{dt^2}I(t) + u(t)I(t) = 0$$

where $u(t)$ depends on the position of the switch. Specifically, suppose, the switch obeys a periodic switching law

$$u(t) = \begin{cases} 1/C_1, & \text{if } 0 \le t < \lambda h; \\ 1/C_2, & \text{if } \lambda h \le t < h, \\ & \text{Extend to be } h\text{-periodic,} \end{cases}$$

where $0 \le \lambda \le 1$ determine the fraction of each duty cycle the switch is closed to the left. Following the methods of the paper, we can write down the *averaged potential* for this system. This is reflected in the evolution equation for the averaged system:

$$\frac{d^2}{dt^2}\bar{I} + (\bar{u}/L + \beta^2/L^2)\bar{I} = 0,$$

where

$$\bar{u} = \frac{\lambda}{C_1} + \frac{1-\lambda}{C_2} \quad \text{and} \quad \beta^2 = \sigma^2 - \bar{v}^2 = \frac{\lambda^2(\lambda-1)^2}{12}h^2(1/C_2 - 1/C_1)^2. \square$$

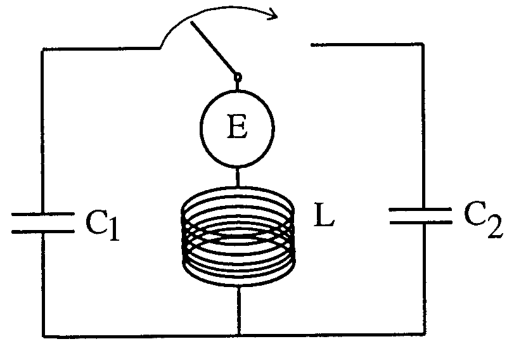

Figure 2: A switched LC-circuit.

In this case, the averaged potential together with the mean of the input reflects the *effective capacitance* of the switched circuit.

5. Concluding Remarks on Connections with Problems in Mechanics

While the theory we have developed applies directly to provide an approach to stability analysis for classical bilinear control systems, and thus applies to models describing electrical networks, it has also been shown in Section 3, that certain models arising Lagrangian mechanics can also be analysed by our averaging method, and in particular stability could be analyzed in terms of an energy-like quantity call the *averaged potential.* In Section 3, we imposed a certain symmetry restriction on the mechanical systems being considered. We conclude by discussing how this restriction can be relaxed.

Remark 6 *(The case of non-symmetric C_i.):* In the case of a system of the form (18), in which the C_i's are not symmetric, we may nevertheless apply the coordinate transformation (8), which in this case is rendered

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ \sum v_i(t) M^{-1} C_i & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

When $S = 0$, the dynamics, written in terms of q-coordinates evolve according to

$$\begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix} = \left[\begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix} + \sum_i v_i \begin{pmatrix} -M^{-1}C_i & 0 \\ 0 & M^{-1}C_i^T \end{pmatrix} + \sum_{i,j} v_i v_j \begin{pmatrix} 0 & 0 \\ M^{-1}C_i^T M^{-1} C_j & 0 \end{pmatrix} \right] \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Averaging the coefficients over one period and applying the "inverse" coordinate transformation,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -\sum \bar v_i M^{-1} C_i & I \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

the "averaged system" dynamics take the form

$$\begin{pmatrix} \dot y_1 \\ \dot y_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}(K + \sum(\sigma_{ij} - \bar v_i \bar v_j) C_i^T M^{-1} C_j) & -M^{-1}(\sum \bar v_i (C_i - C_i^T)) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The quadratic form defining the *averaged potential* for this system is thus seen to be $K + \sum(\sigma_{ij} - \bar{v}_i\bar{v}_j)C_i^T M^{-1} C_j$. If this is positive definite, stability results along the lines of Theorem 2 may be obtained for the case in which a positive definite damping term (associated, say, with a Rayleigh dissipation function as described in Section 3) enters the dynamics. For the case in which there is no dissipation present, the constructions presented in Section 4 are less straightforward, and the stability theory for the case of non-symmetric C_i's remains incomplete. □

Remark 7 *(The relationship with earlier work on the averaged potential.)* In [7], we introduced a more general version of the *averaged potential* aimed at analyzing the motions of a large class of classical mechanical systems. For this larger class of systems, it can be shown that the results of the present paper may be applied directly to linearizations of the dynamics about rest points coinciding with local minima of the averaged potential as introduced in [7]. A geometric averaging theory of mechanical system responses to oscillatory forcing in more general settings (e.g. capable of treating the "hovering" motions identified in [7]) remains to be fully developed. It also remains to explore the role of energy methods in developing a constructive controllability theory along the lines of [10].

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